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Appendix (For Online Publication)

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Financing Local Public Projects

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Equilibrium with debt in the form of n period bonds

Characterization of equilibrium

If

$$\frac{(1-H_0)\overline{\theta}+B(H_0)}{1-\beta} \leq C + \frac{\pi(H_0)}{1-\beta}, \tag{A1}$$

there is no development so that H_{t+1} equals H_0 for all t. The price of building land in each period equals $\pi(H_0)/(1-\beta)$. The price of housing in periods $t \geq n+1$, is equal to

$$P_t = \frac{(1-H_0)\overline{\theta}+B(H_0)}{1-\beta},$$

which is just the present value of the surplus flow from living in the community obtained by the marginal household. Using this and (8), the period t price of housing for $t = 1, ..., n$, is

$$P_t = \frac{(1-H_0)\overline{\theta}+B(H_0)}{1-\beta} - \sum_{z=0}^{n-t} \beta^z \frac{R(D,n)}{H_0}.$$

Note that the future tax payments are capitalized into these prices which implies that prices are increasing as we get closer to the bond being repaid. Moreover, these prices are decreasing in D. Using the expression for P_1 and (8), the period 0 housing price is

$$P_0 = \frac{(1-H_0)\overline{\theta}+\beta B(H_0)}{1-\beta} - \frac{G}{H_0}.$$

This is independent of the debt level, so debt is fully capitalized into the period 0 housing price.

If

$$\frac{(1-H_0)\overline{\theta}+B(H_0)}{1-\beta} > C + \frac{\pi(H_0)}{1-\beta} \geq \frac{(1-H_0)\overline{\theta}+B(H_0) - \frac{R(D,n)}{H_0}}{1-\beta}, \tag{A2}$$

there is development in period $n+1$, but no development before that. Thus, H_{t+1} equals H_0 for all $t \leq n$ and H_{t+1} equals H_{n+2} for all $t > n+1$. The housing level H_{n+2} satisfies

$$\frac{(1-H_{n+2})\overline{\theta}+B(H_{n+2})}{1-\beta} = C + \frac{\pi(H_{n+2})}{1-\beta}.$$

For all periods $t \geq n+1$, the price of building land is $\pi(H_{n+2})/(1-\beta)$ and the price of housing is

$$P_t = C + \frac{\pi(H_{n+2})}{1-\beta}.$$

Housing and building land prices for periods $t = 1, ..., n$ equal

$$P_t = \left((1-H_0)\overline{\theta}+B(H_0) - \frac{R(D,n)}{H_0}\right)\sum_{z=0}^{n-t}\beta^z + \beta^{n+1-t}\left(C + \frac{\pi(H_{n+2})}{1-\beta}\right)$$

and

$$R_t = \pi(H_0)\sum_{z=0}^{n-t}\beta^z + \beta^{n+1-t}\frac{\pi(H_{n+2})}{1-\beta}.$$

Using these expressions and (A2), we have

$$\begin{aligned} P_t - R_t &= \left((1-H_0)\overline{\theta} + B(H_0) - \frac{R(D,n)}{H_0} - \pi(H_0)\right)\sum_{z=0}^{n-t}\beta^z + \beta^{n+1-t}C \\ &\leq C\left((1-\beta)\sum_{z=0}^{n-t}\beta^z + \beta^{n+1-t}\right) = C, \end{aligned}$$

confirming that there is no incentive to develop. Using the expression for P_1 and (8), the period 0 housing price is

$$P_0 = (1-H_0)\overline{\theta} - \frac{G-D}{H_0} + \left((1-H_0)\overline{\theta} + B(H_0) - \frac{R(D,n)}{H_0}\right)\sum_{z=1}^{n}\beta^z + \beta^{n+1}\left(C + \frac{\pi(H_{n+2})}{1-\beta}\right)$$

and the building land price

$$R_0 = \pi(H_0) + \pi(H_0)\sum_{z=1}^{n}\beta^z + \beta^{n+1}\frac{\pi(H_{n+2})}{1-\beta}.$$

Finally, if

$$\frac{(1-H_0)\overline{\theta} + B(H_0) - \frac{R(D,n)}{H_0}}{1-\beta} > C + \frac{\pi(H_0)}{1-\beta}, \tag{A3}$$

there is development in period $n+1$ and development in period 1. Thus, H_1 equals H_0, H_{t+1} equals H_2 for all $t = 1, ..., n$, and H_{t+1} equals H_{n+2} for all $t > n+1$. The housing level H_{n+2} satisfies

$$\frac{(1-H_{n+2})\overline{\theta} + B(H_{n+2})}{1-\beta} = C + \frac{\pi(H_{n+2})}{1-\beta}$$

and the housing level H_2 satisfies

$$\frac{(1-H_2)\overline{\theta} + B(H_2) - \frac{R(D,n)}{H_2}}{1-\beta} = C + \frac{\pi(H_2)}{1-\beta}.$$

For all periods $t \geq n+1$, the price of building land is $\pi(H_{n+2})/(1-\beta)$ and the price of housing is

$$P_t = C + \frac{\pi(H_{n+2})}{1-\beta}.$$

Housing and building land prices for periods $t = 1, ..., n$ equal

$$P_t = \left((1-H_2)\overline{\theta} + B(H_2) - \frac{R(D,n)}{H_2}\right)\sum_{z=0}^{n-t}\beta^z + \beta^{n+1-t}\left(C + \frac{\pi(H_{n+2})}{1-\beta}\right)$$

and

$$R_t = \pi(H_2)\sum_{z=0}^{n-t}\beta^z + \beta^{n+1-t}\frac{\pi(H_{n+2})}{1-\beta}.$$

Using these expressions and (A2), we have

$$\begin{aligned} P_t - R_t &= \left((1-H_2)\overline{\theta} + B(H_2) - \frac{R(D,n)}{H_2} - \pi(H_2)\right)\sum_{z=0}^{n-t}\beta^z + \beta^{n+1-t}C \\ &= C\left([1-\beta]\sum_{z=0}^{n-t}\beta^z + \beta^{n+1-t}\right) = C, \end{aligned}$$

confirming both that there is no incentive to develop further in periods $2, ..., n$ and that H_2 is the equilibrium housing in period 1. In period 0, the price of housing is

$$P_0 = (1-H_0)\overline{\theta} - \frac{G-D}{H_0} + \left((1-H_2)\overline{\theta} + B(H_2) - \frac{R(D,n)}{H_2}\right)\sum_{z=1}^{n}\beta^z + \beta^{n+1}\left(C + \frac{\pi(H_{n+2})}{1-\beta}\right),$$

and the price of building land is

$$R_0 = \pi(H_0) + \pi(H_2)\sum_{z=1}^{n}\beta^z + \beta^{n+1}\frac{\pi(H_{n+2})}{1-\beta}.$$

Equilibrium payoffs

Consider first the period t potential residents for $t \geq n+1$. If θ exceeds $(1-H_{n+2})\overline{\theta}$, such a household purchases a home in the community in period t. In period $t+1$, if they remain in the pool, they enjoy a continuation payoff which is the same as in period t, except that they avoid the cost of buying a house. Given that $P_{t+1} = P_t = P_{n+1}$, it is the case that

$$V_{\theta t} = \theta + B(H_{n+2}) - P_{n+1} + \beta\left[\mu(V_{\theta t} + P_{n+1}) + (1-\mu)P_{n+1}\right].$$

This implies that for all $t \geq n+1$

$$V_{\theta t} = \frac{\theta + B(H_{n+2}) - (1-\beta)P_{n+1}}{1-\mu\beta}. \tag{A4}$$

If θ is less than $(1-H_{n+2})\overline{\theta}$, a period t potential resident never purchases a home in the community and thus their payoff is zero.

Next consider the period t potential residents for $t \in \{1, ..., n\}$. We claim that, if θ exceeds $(1-H_2)\overline{\theta}$, such a household gets

$$\begin{aligned} V_{\theta t} = &\tfrac{\theta + (\mu\beta)^{n+1-t}B(H_{n+2})}{1-\mu\beta} + \left(B(H_2) - \tfrac{R(D,n)}{H_2}\right)\textstyle\sum_{z=0}^{n-t}(\mu\beta)^z - P_t \\ &+(1-\mu)\beta\left[\textstyle\sum_{z=1}^{n-t}(\mu\beta)^{z-1}P_{t+z} + \tfrac{(\mu\beta)^{n-t}P_{n+1}}{1-\mu\beta}\right]. \end{aligned} \tag{A5}$$

To prove this, we first show that the formula is true for $t = n$. We then show that if the formula is true for $t+1$ where $t \in \{1, ..., n-1\}$, it must be true for t.

Consider then a period n potential resident for whom θ exceeds $(1-H_2)\overline{\theta}$. Such a household purchases a home in the community in period n. In period $n+1$, if they remain in the pool, they enjoy a continuation payoff which is equal to $V_{\theta n+1}$ except that they avoid the cost of buying a house. Thus,

$$V_{\theta n} = \theta + B(H_2) - \frac{R(D,n)}{H_2} - P_n + \beta \left[\mu(V_{\theta n+1} + P_{n+1}) + (1-\mu)P_{n+1} \right].$$

Using (A4), we can write this as:

$$V_{\theta n} = \frac{\theta + \mu\beta B(H_{n+2})}{1-\mu\beta} + B(H_2) - \frac{R(D,n)}{H_2} - P_n + \frac{(1-\mu)\beta P_{n+1}}{1-\mu\beta}.$$

This establishes that the formula is true for $t = n$.

Next assume that the formula is true for $t+1$. Consider a period t potential resident for whom θ exceeds $(1-H_2)\overline{\theta}$. Such a household purchases a home in the community in period t. In period $t+1$, if they remain in the pool, they enjoy a continuation payoff which is equal to $V_{\theta t+1}$ except that they avoid the cost of buying a house. Thus,

$$V_{\theta t} = \theta + B(H_2) - \frac{R(D,n)}{H_2} - P_t + \beta \left[\mu(V_{\theta t+1} + P_{t+1}) + (1-\mu)P_{t+1} \right],$$

or equivalently

$$V_{\theta t} = \theta + B(H_2) - \frac{R(D,n)}{H_2} - P_t + \beta\mu V_{\theta t+1} + \beta P_{t+1}.$$

Using the assumption that the formula holds for $t+1$, we can write this as:

$$\begin{aligned} V_{\theta t} &= \theta + B(H_2) - \tfrac{R(D,n)}{H_2} - P_t \\ &+\beta\mu \left(\begin{array}{c} \frac{\theta + (\mu\beta)^{n-t} B(H_{n+2})}{1-\mu\beta} + \left(B(H_2) - \frac{R(D,n)}{H_2} \right) \sum_{z=0}^{n-t-1} (\mu\beta)^z \\ -P_t + (1-\mu)\beta \left[\sum_{z=1}^{n-t-1} (\mu\beta)^{z-1} P_{t+z} + \frac{(\mu\beta)^{n-t-1} P_{n+1}}{1-\mu\beta} \right] \end{array} \right) + \beta P_{t+1} \\ &= \begin{array}{c} \frac{\theta + (\mu\beta)^{n+1-t} B(H_{n+2})}{1-\mu\beta} + \left(B(H_2) - \frac{R(D,n)}{H_2} \right) \sum_{z=0}^{n-t} (\mu\beta)^z - P_t \\ +(1-\mu)\beta \left[\sum_{z=1}^{n-t} (\mu\beta)^{z-1} P_{t+z} + \frac{(\mu\beta)^{n-t} P_{n+1}}{1-\mu\beta} \right] \end{array} \end{aligned}$$

This establishes that the formula is true for t if it is true for $t+1$. We conclude that (A5) holds.

Continuing with the period t potential residents, if θ is between $(1-H_{n+2})\overline{\theta}$ and $(1-H_2)\overline{\theta}$ such a household purchases a home in the community in period $n+1$ if they remain in the pool at that time and remain there as long as they are in the pool. If they remain in the pool in period $n+1$, they enjoy a continuation payoff equal to $V_{\theta n+1}$. Thus, $V_{\theta t}$ is equal to $(\beta\mu)^{n+1-t}\, V_{\theta n+1}$. If θ is less than $(1-H_{n+2})\overline{\theta}$, such a household has a payoff of 0.

Next consider the period 0 potential residents. If θ exceeds $(1-H_0)\overline{\theta}$, such a household purchases a home in the community in period 0. By the usual argument

$$V_{\theta 0} = \theta - \frac{G-D}{H_0} - P_0 + \beta\mu V_{\theta 1} + \beta P_1.$$

Using (A5), we can write this as:

$$\begin{aligned} V_{\theta 0} = {} & \tfrac{\theta + (\mu\beta)^{n+1} B(H_{n+2})}{1-\mu\beta} + \left(B(H_2) - \tfrac{R(D,n)}{H_2}\right) \textstyle\sum_{z=1}^{n} (\mu\beta)^z - \tfrac{G-D}{H_0} - P_0 \\ & + (1-\mu)\beta \left[\textstyle\sum_{z=0}^{n-1} (\mu\beta)^z P_{1+z} + \tfrac{(\mu\beta)^n P_{n+1}}{1-\mu\beta}\right]. \end{aligned} \tag{A6}$$

If θ is between $(1-H_2)\overline{\theta}$ and $(1-H_0)\overline{\theta}$ a period 0 potential resident purchases a home in the community in period 1. If they remain in the pool in period 1, they enjoy a continuation payoff equal to $V_{\theta 1}$. Thus, $V_{\theta 0}$ is equal to $\beta\mu V_{\theta 1}$. If θ is between $(1-H_{n+2})\overline{\theta}$ and $(1-H_2)\overline{\theta}$ a period 0 potential resident purchases a home in the community in period $n+1$ if they remain in the pool at that time. If they remain in the pool in period $n+1$, they enjoy a continuation payoff equal to $V_{\theta n+1}$. Thus, $V_{\theta 0}$ is equal to $(\beta\mu)^{n+1}\, V_{\theta n+1}$. If θ is less than $(1-H_{n+1})\overline{\theta}$, a period 0 potential resident obtains a zero payoff.

The final cohort of residents are the initial potential residents. The initial residents; i.e., those for whom θ exceeds $(1-H_0)\overline{\theta}$, own a home in the community and remain there as long as they are in the pool. If they remain in the pool in period 0, they enjoy a payoff equal to $V_{\theta 0}$ except that they avoid the cost of buying a house. Thus,

$$V_\theta = \mu V_{\theta 0} + P_0. \tag{A7}$$

The initial potential residents for whom θ is between $(1-H_2)\overline{\theta}$ and $(1-H_0)\overline{\theta}$ do not own a home but purchase a home in the community in period 1 if they remain in the pool. In this case, they enjoy a continuation payoff equal to $V_{\theta 1}$. Thus, $V_\theta = \mu\beta\mu V_{\theta 1}$. If θ is between $(1-H_{n+2})\overline{\theta}$ and $(1-H_2)\overline{\theta}$ an initial period potential resident purchases a home in the community in period $n+1$ if they remain in the pool at that time. In this case, they enjoy a continuation payoff equal to $V_{\theta n+1}$. Thus, $V_{\theta 0}$ is equal to $\mu\,(\beta\mu)^{n+1}\, V_{\theta n+1}$. If θ is less than $(1-H_{n+1})\overline{\theta}$, a initial period potential resident obtains a zero payoff.

Finally, consider the landowners. Those with land of productivity π between $\pi(H_0)$ and $\pi(H_2)$ sell their land in period 1 and obtain a payoff of $\pi+\beta R_1$. Those with land of productivity π between $\pi(H_2)$ and $\pi(H_{n+2})$ sell their land in period $n+1$ and obtain a payoff of $\pi \sum_{z=0}^{n} \beta^z + \beta^{n+1} R_{n+1}$. Landowners with land of higher productivity never sell their land for building and just obtain a payoff $\pi/(1-\beta)$.

Equilibrium with an n period development tax

Characterization of equilibrium

As in Section 4, we restrict attention to development taxes satisfying $\tau \leq G/\beta H_0$. If

$$\frac{(1-H_0)\overline{\theta}+B(H_0)}{1-\beta} \leq C + \frac{\pi(H_0)}{1-\beta}, \tag{A8}$$

there is no development so that H_{t+1} equals H_0 for all t. The price of building land in each period equals $\pi(H_0)/(1-\beta)$. The price of housing in periods $t \geq 1$ is constant and equal to

$$P_t = \frac{(1-H_0)\overline{\theta}+B(H_0)}{1-\beta}.$$

Using this and (8), the period 0 housing price is

$$P_0 = \frac{(1-H_0)\overline{\theta}+\beta B(H_0)}{1-\beta} - \frac{G}{H_0}.$$

If

$$C + \frac{\pi(H_0)}{1-\beta} + \frac{\tau}{\sum_{z=0}^{n-1} \beta^z (1-\beta)} \geq \frac{(1-H_0)\overline{\theta}+B(H_0)}{1-\beta} > C + \frac{\pi(H_0)}{1-\beta}, \tag{A9}$$

there will be development in period $n+1$. Housing level H_{n+2} will satisfy

$$\frac{(1-H_{n+2})\overline{\theta}+B(H_{n+2})}{1-\beta} = C + \frac{\pi(H_{n+2})}{1-\beta}$$

and for all $t \geq n+2$, $H_t = H_{n+2}$. For all $t \geq n+1$, the price of housing is

$$P_t = C + \frac{\pi(H_{n+2})}{1-\beta}$$

and the price of building land is $\pi(H_{n+2})/(1-\beta)$.

In periods 1 through n, the prices of housing and building land are

$$P_t = \left((1-H_0)\overline{\theta}+B(H_0)\right) \sum_{z=0}^{n-t} \beta^z + \beta^{n+1-t} \left(C + \frac{\pi(H_{n+2})}{1-\beta} \right),$$

and

$$R_t = \pi(H_0)\sum_{z=0}^{n-t}\beta^z + \beta^{n+1-t}\frac{\pi(H_{n+2})}{1-\beta}.$$

Using (A9), we have that

$$\begin{aligned}
P_t - R_t &= \big((1-H_0)\overline{\theta} + B(H_0) - \pi(H_0)\big)\sum_{z=0}^{n-t}\beta^z + \beta^{n+1-t}C \\
&\leq \left(C(1-\beta) + \frac{\tau}{\sum_{z=0}^{n-1}\beta^z}\right)\sum_{z=0}^{n-t}\beta^z + \beta^{n+1-t}C \\
&= C + \tau\frac{\sum_{z=0}^{n-t}\beta^z}{\sum_{z=0}^{n-1}\beta^z} \leq C + \tau.
\end{aligned}$$

Thus, there is no incentive for new construction in these periods. Furthermore, note that

$$\begin{aligned}
P_t - P_{t-1} &= \big((1-H_0)\overline{\theta} + B(H_0)\big)\sum_{z=0}^{n-t}\beta^z + \beta^{n+1-t}\left(C + \frac{\pi(H_{n+2})}{1-\beta}\right) \\
&\quad - \left[\big((1-H_0)\overline{\theta} + B(H_0)\big)\sum_{z=0}^{n-(t-1)}\beta^z + \beta^{n+1-(t-1)}\left(C + \frac{\pi(H_{n+2})}{1-\beta}\right)\right] \\
&= -\big((1-H_0)\overline{\theta} + B(H_0)\big)\beta^{n-t+1} + \beta^{n+1-t}(1-\beta)\left(C + \frac{\pi(H_{n+2})}{1-\beta}\right) \\
&= \beta^{n+1-t}\left[C(1-\beta) + \pi(H_{n+2}) - \big((1-H_0)\overline{\theta} + B(H_0)\big)\right] \\
&< \beta^{n+1-t}\left[C(1-\beta) + \pi(H_{n+2}) - \big((1-H_{n+2})\overline{\theta} + B(H_{n+2})\big)\right] = 0
\end{aligned}$$

Thus, housing prices are decreasing as we approach period $n+1$.

In period 0, the prices of housing and building land are

$$P_0 = (1-H_0)\overline{\theta} - \frac{G}{H_0} + \big((1-H_0)\overline{\theta} + B(H_0)\big)\sum_{z=1}^{n}\beta^z + \beta^{n+1}\left(C + \frac{\pi(H_{n+2})}{1-\beta}\right),$$

and

$$R_0 = \pi(H_0)\sum_{z=0}^{n}\beta^z + \beta^{n+1}\frac{\pi(H_{n+2})}{1-\beta}.$$

From (A9), (2), and the assumption that $\tau \leq G/\beta H_0$, we have that

$$P_0 - R_0 = (1-H_0)\overline{\theta} - \frac{G}{H_0} - \pi(H_0) + \big((1-H_0)\overline{\theta} + B(H_0) - \pi(H_0)\big)\sum_{z=1}^{n}\beta^z + \beta^{n+1}C$$

$$\begin{aligned}
&\leq (1-H_0)\overline{\theta} - \frac{G}{H_0} - \pi(H_0) + \left(C(1-\beta) + \frac{\tau}{\sum_{z=0}^{n-1}\beta^z} \right) \sum_{z=1}^{n}\beta^z + \beta^{n+1}C \\
&= (1-H_0)\overline{\theta} - \frac{G}{H_0} - \pi(H_0) + C\left((1-\beta)\sum_{z=1}^{n}\beta^z + \beta^{n+1} \right) + \tau\frac{\sum_{z=1}^{n}\beta^z}{\sum_{z=0}^{n-1}\beta^z} \\
&= (1-H_0)\overline{\theta} - \frac{G}{H_0} - \pi(H_0) + C\beta + \tau\beta \\
&\leq (1-H_0)\overline{\theta} - \pi(H_0) + C\beta \leq C
\end{aligned}$$

Thus, there is no incentive for development in period 0.

If

$$\frac{(1-H_0)\overline{\theta} + B(H_0)}{1-\beta} > C + \frac{\pi(H_0)}{1-\beta} + \frac{\tau}{\sum_{z=0}^{n-1}\beta^z(1-\beta)}, \tag{A10}$$

there will be development in period $n+1$ and period 1. Housing level H_{n+2} will satisfy

$$\frac{(1-H_{n+2})\overline{\theta} + B(H_{n+2})}{1-\beta} = C + \frac{\pi(H_{n+2})}{1-\beta}$$

and for all $t \geq n+2$, $H_t = H_{n+2}$. For all $t \geq n+1$, the price of housing is

$$P_t = C + \frac{\pi(H_{n+2})}{1-\beta}$$

and the price of building land is $\pi(H_{n+2})/(1-\beta)$.

Housing level H_2 satisfies

$$(1-H_2)\overline{\theta} + B(H_2) = C(1-\beta) + \pi(H_2) + \frac{\tau H_0}{\sum_{z=0}^{n-1}\beta^z H_2}. \tag{A11}$$

and for all $t = 1, ..., n$, $H_{t+1} = H_2$. In periods $t = 2, ..., n$ the price of housing is

$$P_t = \left((1-H_2)\overline{\theta} + B(H_2)\right)\sum_{z=0}^{n-t}\beta^z + \beta^{n+1-t}\left(C + \frac{\pi(H_{n+2})}{1-\beta}\right),$$

and the price of building land is

$$R_t = \pi(H_2)\sum_{z=0}^{n-t}\beta^z + \beta^{n+1-t}\frac{\pi(H_{n+2})}{1-\beta}.$$

Using (A11), we have that

$$\begin{aligned} P_t - R_t &= \big((1-H_2)\overline{\theta} + B(H_2) - \pi(H_2)\big)\sum_{z=0}^{n-t}\beta^z + \beta^{n+1-t}C \\ &= \left(C(1-\beta) + \frac{\tau}{\sum_{z=0}^{n-1}\beta^z}\right)\sum_{z=0}^{n-t}\beta^z + \beta^{n+1-t}C \\ &= C + \tau\frac{\sum_{z=0}^{n-t}\beta^z}{\sum_{z=0}^{n-1}\beta^z} < C + \tau. \end{aligned}$$

Thus, there is no incentive for new construction in periods 2 through n. Furthermore,

$$\begin{aligned} P_t - P_{t-1} &= \big((1-H_2)\overline{\theta} + B(H_2)\big)\sum_{z=0}^{n-t}\beta^z + \beta^{n+1-t}\left(C + \frac{\pi(H_{n+2})}{1-\beta}\right) \\ &\quad - \left[\big((1-H_2)\overline{\theta} + B(H_2)\big)\sum_{z=0}^{n-(t-1)}\beta^z + \beta^{n+1-(t-1)}\left(C + \frac{\pi(H_{n+2})}{1-\beta}\right)\right] \\ &= \beta^{n+1-t}\left[C(1-\beta) + \pi(H_{n+2}) - \big((1-H_2)\overline{\theta} + B(H_2)\big)\right] \\ &< \beta^{n+1-t}\left[C(1-\beta) + \pi(H_{n+2}) - \big((1-H_{n+2})\overline{\theta} + B(H_{n+2})\big)\right] = 0. \end{aligned}$$

Thus, housing prices are decreasing as we approach period $n+1$.

In period 1, the prices of housing and building land are

$$P_1 = C + \pi(H_2)\sum_{z=0}^{n-1}\beta^z + \beta^n\frac{\pi(H_{n+2})}{1-\beta} + \tau$$

and

$$R_1 = \pi(H_2)\sum_{z=0}^{n-1}\beta^z + \beta^n\frac{\pi(H_{n+2})}{1-\beta}.$$

The solution for the housing level H_2 provided in (A11) follows from solving the market clearing condition

$$\big((1-H_2)\overline{\theta} + B(H_2)\big)\sum_{z=0}^{n-1}\beta^z + \tau\frac{H_2 - H_0}{H_2} + \beta^n\left(C + \frac{\pi(H_{n+2})}{1-\beta}\right) = C + \pi(H_2)\sum_{z=0}^{n-1}\beta^z + \beta^n\frac{\pi(H_{n+2})}{1-\beta} + \tau.$$

In period 0, the prices of housing and building land are

$$P_0 = (1-H_0)\overline{\theta} - \frac{G}{H_0} + \beta\left(C + \pi(H_2)\sum_{z=0}^{n-1}\beta^z + \beta^n\frac{\pi(H_{n+2})}{1-\beta} + \tau\right),$$

and

$$R_0 = \pi(H_0) + \beta\left(\pi(H_2)\sum_{z=0}^{n-1}\beta^z + \beta^n\frac{\pi(H_{n+2})}{1-\beta}\right).$$

Note that, using the assumption that $\tau \leq G/\beta H_0$ and (2), we have that

$$\begin{aligned} P_0 - R_0 = (1-H_0)\overline{\theta} - \tfrac{G}{H_0} + \beta\left(C + \pi(H_2)\sum_{z=0}^{n-1}\beta^z + \beta^n\tfrac{\pi(H_{n+2})}{1-\beta} + \tau\right) \\ -\pi(H_0) - \beta\left(\pi(H_2)\sum_{z=0}^{n-1}\beta^z + \beta^n\tfrac{\pi(H_{n+2})}{1-\beta}\right) \\ = \quad (1-H_0)\overline{\theta} - \frac{G}{H_0} + \beta C + \beta\tau - \pi(H_0) \\ \leq \quad (1-H_0)\overline{\theta} + \beta C - \pi(H_0) \\ \leq \quad C. \end{aligned}$$

Thus, there is no incentive for development in period 0.

Equilibrium payoffs

Regarding the period t potential residents for $t \geq n+1$, the logic from the equilibrium with debt applies. Thus, if θ exceeds $(1-H_{n+2})\overline{\theta}$, it is the case that

$$V_{\theta t} = \frac{\theta + B(H_{n+2}) - (1-\beta)P_{n+1}}{1-\mu\beta}. \tag{A12}$$

If θ is less than $(1-H_{n+2})\overline{\theta}$, a period t potential resident obtains a payoff of zero.

Next consider the period $t = 2, ..., n$ potential residents. We claim that, if θ exceeds $(1-H_2)\overline{\theta}$, such a household gets

$$V_{\theta t} = \frac{\theta + (\mu\beta)^{n+1-t}B(H_{n+2})}{1-\mu\beta} + B(H_2)\sum_{z=0}^{n-t}(\mu\beta)^z - P_t + (1-\mu)\beta\left[\sum_{z=1}^{n-t}(\mu\beta)^{z-1}P_{t+z} + \frac{(\mu\beta)^{n-t}P_{n+1}}{1-\mu\beta}\right]. \tag{A13}$$

To prove this, we first show that the formula is true for $t = n$. We then show that if the formula is true for $t+1$ where $t \in \{1, ..., n-1\}$, it must be true for t.

Consider then a period n potential resident for whom θ exceeds $(1-H_2)\overline{\theta}$. Such a household purchases a home in the community in period n. In period $n+1$, if they remain in the pool, they enjoy a continuation payoff which is equal to $V_{\theta n+1}$ except that they avoid the cost of buying a house. Thus,

$$V_{\theta n} = \theta + B(H_2) - P_n + \beta\left[\mu(V_{\theta n+1} + P_{n+1}) + (1-\mu)P_{n+1}\right].$$

Using (A12), we can write this as:

$$V_{\theta n} = \frac{\theta + \mu\beta B(H_{n+2})}{1-\mu\beta} + B(H_2) - P_n + (1-\mu)\beta\left(\frac{P_{n+1}}{1-\mu\beta}\right).$$

This establishes that the formula is true for $t = n$.

Next assume that the formula is true for $t+1$. Consider a period t potential resident for whom θ exceeds $(1-H_2)\overline{\theta}$. Such a household purchases a home in the community in period t. In period $t+1$, if they remain in the pool, they enjoy a continuation payoff which is equal to $V_{\theta t+1}$ except that they avoid the cost of buying a house. Thus,

$$V_{\theta t} = \theta + B(H_2) - P_t + \beta\mu V_{\theta t+1} + \beta P_{t+1}.$$

Using the assumption that the formula holds for $t+1$, we can write this as:

$$\begin{aligned} V_{\theta t} &= \theta + B(H_2) - P_t + \beta\mu \left(\begin{array}{c} \frac{\theta+(\mu\beta)^{n-t}B(H_{n+2})}{1-\mu\beta} + B(H_2)\sum_{z=0}^{n-t-1}(\mu\beta)^z \\ -P_t + (1-\mu)\beta\left[\sum_{z=1}^{n-t-1}(\mu\beta)^{z-1}P_{t+z} + \frac{(\mu\beta)^{n-t-1}P_{n+1}}{1-\mu\beta}\right] \end{array} \right) + \beta P_{t+1} \\ &= \frac{\theta + (\mu\beta)^{n+1-t}B(H_{n+2})}{1-\mu\beta} + B(H_2)\sum_{z=0}^{n-t}(\mu\beta)^z - P_t + (1-\mu)\beta\left[\sum_{z=1}^{n-t}(\mu\beta)^{z-1}P_{t+z} + \frac{(\mu\beta)^{n-t}P_{n+1}}{1-\mu\beta}\right] \end{aligned}$$

This establishes that the formula is true for t if it is true for $t+1$. We conclude that (A13) holds.

Continuing with the period t potential residents, if θ is between $(1-H_{n+2})\overline{\theta}$ and $(1-H_2)\overline{\theta}$ such a household purchases a home in the community in period $n+1$ if they remain in the pool at that time. In this case, they enjoy a continuation payoff equal to $V_{\theta n+1}$. Thus, $V_{\theta t}$ is equal to $(\beta\mu)^{n+1-t}V_{\theta n+1}$. If θ is less than $(1-H_{n+2})\overline{\theta}$, such a household has a payoff of 0.

Next consider the period 1 potential residents. If θ exceeds $(1-H_2)\overline{\theta}$, such a household purchases a home in the community in period 1. In period 2, if they remain in the pool, they enjoy a continuation payoff which is equal to $V_{\theta 2}$ except that they avoid the cost of buying a house. Thus,

$$V_{\theta 1} = \theta + B(H_2) + \tau\frac{(H_2 - H_0)}{H_2} - P_1 + \beta\left[\mu(V_{\theta 2} + P_2) + (1-\mu)P_2\right].$$

Using (A13), we can write this as:

$$\begin{aligned} V_{\theta 1} = \frac{\theta+(\mu\beta)^n B(H_{n+2})}{1-\mu\beta} + \tau\frac{(H_2-H_0)}{H_2} - P_1 \\ +B(H_2)\sum_{z=0}^{n-1}(\mu\beta)^z + (1-\mu)\beta\left[\sum_{z=1}^{n-1}(\mu\beta)^{z-1}P_{1+z} + \frac{(\mu\beta)^{n-1}P_{n+1}}{1-\mu\beta}\right]. \end{aligned} \tag{A14}$$

If θ is between $(1-H_{n+2})\overline{\theta}$ and $(1-H_2)\overline{\theta}$ a period 1 potential resident purchases a home in the community in period $n+1$ if they remain in the pool at that time. In thsi case, they enjoy a continuation payoff equal to $V_{\theta n+1}$. Thus, $V_{\theta 1}$ is equal to $(\beta\mu)^n V_{\theta n+1}$.

Next consider the period 0 potential residents. If θ exceeds $(1-H_0)\overline{\theta}$, such a household purchases a home in the community in period 0. In period 1, if they remain in the pool, they enjoy a continuation payoff which is equal to $V_{\theta 1}$ except that they avoid the cost of buying a house. Thus,

$$V_{\theta 0} = \theta - \frac{G}{H_0} - P_0 + \beta\left[\mu(V_{\theta 1} + P_1) + (1-\mu)P_1\right].$$

Using (A14), we can write this as:

$$\begin{aligned} V_{\theta 0} = {} & \tfrac{\theta + (\mu\beta)^{n+1} B(H_{n+2})}{1-\mu\beta} + B(H_2)\textstyle\sum_{z=1}^{n}(\mu\beta)^z + \beta\mu\tau\tfrac{(H_2-H_0)}{H_2} - \tfrac{G}{H_0} - P_0 \\ & + (1-\mu)\beta\left[\textstyle\sum_{z=0}^{n-1}(\mu\beta)^z P_{1+z} + \tfrac{(\mu\beta)^n P_{n+1}}{1-\mu\beta}\right]. \end{aligned} \tag{A15}$$

If θ is between $(1-H_2)\overline{\theta}$ and $(1-H_0)\overline{\theta}$ a period 0 potential resident purchases a home in the community in period 1 if they remain in the pool at that time. In this case, they enjoy a continuation payoff equal to $V_{\theta 1}$. Thus, $V_{\theta 0}$ is equal to $\beta\mu V_{\theta 1}$. If θ is between $(1-H_{n+2})\overline{\theta}$ and $(1-H_2)\overline{\theta}$ a period 0 potential resident purchases a home in the community in period $n+1$ if they remain in the pool at that time. In this case, they enjoy a continuation payoff equal to $V_{\theta n+1}$. Thus, $V_{\theta 0}$ is equal to $(\beta\mu)^{n+1} V_{\theta n+1}$. If θ is less than $(1-H_{n+1})\overline{\theta}$, a period 0 potential resident obtains a zero payoff.

The final cohort of residents are the initial potential residents. The initial residents; i.e., those for whom θ exceeds $(1-H_0)\overline{\theta}$, own a home in the community. If they remain in the pool in period 0, they enjoy a payoff equal to $V_{\theta 0}$ except that they avoid the cost of buying a house. Thus,

$$V_\theta = \mu\left(V_{\theta 0} + P_0\right) + (1-\mu)P_0 = \mu V_{\theta 0} + P_0.$$

The initial potential residents for whom θ is between $(1-H_2)\overline{\theta}$ and $(1-H_0)\overline{\theta}$ do not own a home but purchase a home in the community in period 1 if they remain in the pool. In this case, they enjoy a continuation payoff equal to $V_{\theta 1}$. Thus, $V_\theta = \mu\beta\mu V_{\theta 1}$. If θ is between $(1-H_{n+2})\overline{\theta}$ and $(1-H_2)\overline{\theta}$ an initial period potential resident purchases a home in the community in period $n+1$ if they remain in the pool at that time. In this case, they enjoy a continuation payoff equal to $V_{\theta n+1}$. Thus, $V_{\theta 0}$ is equal to $\mu\left(\beta\mu\right)^{n+1} V_{\theta n+1}$. If θ is less than $(1-H_{n+1})\overline{\theta}$, a initial period potential resident obtains a zero payoff.

Finally, consider the landowners. Those with land of productivity π between $\pi(H_0)$ and $\pi(H_2)$ sell their land in period 1 and obtain a payoff of $\pi+\beta R_1$. Those with land of productivity π between $\pi(H_2)$ and $\pi(H_{n+2})$ sell their land in period $n+1$ and obtain a payoff of $\pi\sum_{z=0}^{n}\beta^z+\beta^{n+1}R_{n+1}$. Landowners with land of higher productivity never sell their land for building and just obtain a payoff $\pi/(1-\beta)$.

Proof of Proposition 5

Let $\{H_{t+1}^*, P_t^*, R_t^*\}_{t=0}^{\infty}$ be an equilibrium with debt D in the form of n period bonds. We need to show that $\{H_{t+1}^*, \widetilde{P}_t, R_t^*\}_{t=0}^{\infty}$ is an equilibrium with n period development tax τ^* where the tax τ^* and price sequence $\{\widetilde{P}_t\}_{t=0}^{\infty}$ are as defined in the statement of the Proposition. There are three possibilities to consider: the equilibrium with debt involves development in periods 1 and $n+1$ ($H_2^* > H_0$), development in only period $n+1$ ($H_2^* = H_0$ and $H_{n+2}^* > H_0$), or no development ($H_{n+2}^* = H_0$). We consider each in turn.

Suppose first that there is no development. Then we know that

$$\frac{(1-H_0)\overline{\theta}+B(H_0)}{1-\beta} \leq C + \frac{\pi(H_0)}{1-\beta}. \tag{A16}$$

Moreover, H_{t+1}^* equals H_0 for all t. The price of building land in each period equals $\pi(H_0)/(1-\beta)$. The price of housing in periods $t \geq n+1$, is equal to

$$P_t^* = \frac{(1-H_0)\overline{\theta}+B(H_0)}{1-\beta}.$$

For periods $t = 1, ..., n$

$$P_t^* = \frac{(1-H_0)\overline{\theta}+B(H_0)}{1-\beta} - \sum_{z=0}^{n-t}\beta^z\frac{R(D,n)}{H_0}.$$

For period 0, we have

$$P_0^* = \frac{(1-H_0)\overline{\theta}+\beta B(H_0)}{1-\beta} - \frac{G}{H_0}.$$

To show that $\{H_{t+1}^*, \widetilde{P}_t, R_t^*\}_{t=0}^{\infty}$ is an equilibrium with n period development tax τ^* where the tax τ^* and price sequence $\{\widetilde{P}_t\}_{t=0}^{\infty}$ are as defined in the statement of the Proposition, the only thing to verify in this case is that the housing prices are equilibrium prices. This requires that in periods $t = 1, ..., n$

$$\widetilde{P}_t = \frac{(1-H_0)\overline{\theta}+B(H_0)}{1-\beta}.$$

But we know that for $t = 2, ..., n$, since $H_2^* = H_0$, we have that

$$\begin{aligned}\widetilde{P}_t &= P_t^* + \frac{\tau^* \sum_{z=1}^{n+1-t} \beta^z}{\sum_{z=1}^{n} \beta^z} \\ &= \frac{(1-H_0)\overline{\theta} + B(H_0)}{1-\beta} - \sum_{z=0}^{n-t} \beta^z \frac{R(D,n)}{H_0} + \frac{\tau^* \sum_{z=1}^{n+1-t} \beta^z}{\sum_{z=1}^{n} \beta^z}.\end{aligned}$$

Thus, we need to show that

$$\frac{\tau^* \sum_{z=1}^{n+1-t} \beta^z}{\sum_{z=1}^{n} \beta^z} = \sum_{z=0}^{n-t} \beta^z \frac{R(D,n)}{H_0}.$$

We have

$$\frac{\tau^* \sum_{z=1}^{n+1-t} \beta^z}{\sum_{z=1}^{n} \beta^z} = \frac{D \sum_{z=1}^{n+1-t} \beta^z}{H_0 \sum_{z=1}^{n} \beta^{z+1}}$$

and

$$\sum_{z=0}^{n-t} \beta^z \frac{R(D,n)}{H_0} = \frac{\sum_{z=0}^{n-t} \beta^z D}{H_0 \sum_{z=1}^{n} \beta^z} = \frac{\beta \sum_{z=0}^{n-t} \beta^z D}{H_0 \beta \sum_{z=1}^{n} \beta^z} = \frac{\sum_{z=0}^{n-t} \beta^{z+1} D}{H_0 \sum_{z=1}^{n} \beta^{z+1}} = \frac{D \sum_{w=1}^{n+1-t} \beta^w}{H_0 \sum_{z=1}^{n} \beta^{z+1}}.$$

Thus, the desired equality holds. Moreover, for period 1 we have that

$$\widetilde{P}_1 = P_1^* + \tau^* = \frac{(1-H_0)\overline{\theta} + B(H_0)}{1-\beta} - \sum_{z=0}^{n-1} \beta^z \frac{R(D,n)}{H_0} + \tau^*.$$

Thus, we need to show that

$$\tau^* = \sum_{z=0}^{n-1} \beta^z \frac{R(D,n)}{H_0}.$$

We have

$$\tau^* = \frac{D}{\beta H_0}$$

and

$$\sum_{z=0}^{n-1} \beta^z \frac{R(D,n)}{H_0} = \frac{\sum_{z=0}^{n-1} \beta^z D}{H_0 \sum_{z=1}^{n} \beta^z} = \frac{\beta \sum_{z=0}^{n-1} \beta^z D}{H_0 \beta \sum_{z=1}^{n} \beta^z} = \frac{\sum_{z=0}^{n-1} \beta^{z+1} D}{H_0 \sum_{z=1}^{n} \beta^{z+1}} = \frac{D \sum_{w=1}^{n} \beta^w}{H_0 \sum_{z=1}^{n} \beta^{z+1}} = \frac{D}{\beta H_0}.$$

Next suppose that there is only development in period $n+1$. Then we know that

$$\frac{(1-H_0)\overline{\theta} + B(H_0)}{1-\beta} > C + \frac{\pi(H_0)}{1-\beta} \geq \frac{(1-H_0)\overline{\theta} + B(H_0) - \frac{R(D,n)}{H_0}}{1-\beta}. \tag{A17}$$

The housing level H_{n+2}^* will satisfy

$$\frac{(1-H_{n+2}^*)\overline{\theta} + B(H_{n+2}^*)}{1-\beta} = C + \frac{\pi(H_{n+2}^*)}{1-\beta}$$

and for all $t \geq n+2$, $H_t^* = H_{n+2}^*$. In addition, for all $t \geq n+1$

$$P_t^* = C + \frac{\pi(H_{n+2}^*)}{1-\beta}.$$

Furthermore, H_{t+1}^* equals H_0 for all $t = 0, ..., n$. Prices for periods $t = 1, ..., n$ equal

$$\begin{aligned} P_t^* &= \left((1-H_0)\overline{\theta} + B(H_0) - \frac{R(D,n)}{H_0}\right)\sum_{z=0}^{n-t}\beta^z + \beta^{n+1-t}\left(C + \frac{\pi(H_{n+2}^*)}{1-\beta}\right) \\ R_t &= \pi(H_0)\sum_{z=0}^{n-t}\beta^z + \beta^{n+1-t}\frac{\pi(H_{n+2}^*)}{1-\beta}. \end{aligned}$$

Period 0 prices are

$$\begin{aligned} P_0 &= (1-H_0)\overline{\theta} - \frac{G-D}{H_0} + \left((1-H_0)\overline{\theta} + B(H_0) - \frac{R(D,n)}{H_0}\right)\sum_{z=1}^{n}\beta^z + \beta^{n+1}\left(C + \frac{\pi(H_{n+2}^*)}{1-\beta}\right) \\ R_0 &= \pi(H_0) + \pi(H_0)\sum_{z=1}^{n}\beta^z + \beta^{n+1}\frac{\pi(H_{n+2}^*)}{1-\beta}. \end{aligned}$$

To show that $\{H_{t+1}^*, \widetilde{P}_t, R_t^*\}_{t=0}^{\infty}$ is an equilibrium with n period development tax τ^* where the tax τ^* and price sequence $\{\widetilde{P}_t\}_{t=0}^{\infty}$ are as defined in the statement of the Proposition, we first need to verify that there is only development in period $n+1$. This requires that

$$\frac{(1-H_0)\overline{\theta} + B(H_0)}{1-\beta} > C + \frac{\pi(H_0)}{1-\beta} \geq \frac{(1-H_0)\overline{\theta} + B(H_0) - \tau^*/\sum_{z=0}^{n-1}\beta^z}{1-\beta}.$$

This follows from the fact that

$$\tau^*/\sum_{z=0}^{n-1}\beta^z = D/H_0\sum_{z=0}^{n-1}\beta^{z+1} = D/H_0\sum_{z=1}^{n}\beta^z = \frac{R(D,n)}{H_0}.$$

In terms of prices, we require that in periods $t = 1, ..., n$

$$\widetilde{P}_t = \left((1-H_0)\overline{\theta} + B(H_0)\right)\sum_{z=0}^{n-t}\beta^z + \beta^{n+1-t}\left(C + \frac{\pi(H_{n+2}^*)}{1-\beta}\right).$$

But we know that for $t = 1, ..., n$, since $H_2^* = H_0$, we have that

$$\begin{aligned} \widetilde{P}_t &= P_t^* + \frac{\tau^*\sum_{z=1}^{n+1-t}\beta^z}{\sum_{z=1}^{n}\beta^z} \\ &= \left((1-H_0)\overline{\theta} + B(H_0) - \frac{R(D,n)}{H_0}\right)\sum_{z=0}^{n-t}\beta^z + \beta^{n+1-t}\left(C + \frac{\pi(H_{n+2}^*)}{1-\beta}\right) + \frac{\tau^*\sum_{z=1}^{n+1-t}\beta^z}{\sum_{z=1}^{n}\beta^z}. \end{aligned}$$

Thus, we need to show that

$$\frac{\tau^*\sum_{z=1}^{n+1-t}\beta^z}{\sum_{z=1}^{n}\beta^z} = \sum_{z=0}^{n-t}\beta^z\frac{R(D,n)}{H_0},$$

which is true. Moreover, for period 0, we require that

$$\widetilde{P}_0 = (1-H_0)\overline{\theta} - \frac{G}{H_0} + \left((1-H_0)\overline{\theta} + B(H_0)\right)\sum_{z=1}^{n}\beta^z + \beta^{n+1}\left(C + \frac{\pi(H^*_{n+2})}{1-\beta}\right).$$

We have

$$\widetilde{P}_0 = P^*_0 = (1-H_0)\overline{\theta} - \frac{G-D}{H_0} + \left((1-H_0)\overline{\theta} + B(H_0) - \frac{R(D,n)}{H_0}\right)\sum_{z=1}^{n}\beta^z + \beta^{n+1}\left(C + \frac{\pi(H^*_{n+2})}{1-\beta}\right).$$

Thus, we need to show that

$$\frac{D}{H_0} = \sum_{z=1}^{n}\beta^z \frac{R(D,n)}{H_0},$$

which is true.

Finally suppose that there is development in period 1. Then we know that

$$C + \frac{\pi(H_0)}{1-\beta} < \frac{(1-H_0)\overline{\theta} + B(H_0) - \frac{R(D,n)}{H_0}}{1-\beta}. \tag{A18}$$

Housing level H^*_{n+2} will satisfy

$$\frac{(1-H^*_{n+2})\overline{\theta} + B(H^*_{n+2})}{1-\beta} = C + \frac{\pi(H^*_{n+2})}{1-\beta}$$

and for all $t \geq n+2$, $H^*_t = H^*_{n+2}$. For all $t \geq n+1$

$$P^*_t = C + \frac{\pi(H^*_{n+2})}{1-\beta}.$$

In addition, H^*_{t+1} equals H^*_2 for all $t = 1, ..., n$, where

$$(1-H^*_2)\overline{\theta} + B(H^*_2) - \frac{R(D,n)}{H^*_2} = C(1-\beta) + \pi(H^*_2).$$

The prices of housing and building land in periods $t = 1, ..., n$ are as follows:

$$\begin{aligned} P^*_t &= \left((1-H^*_2)\overline{\theta} + B(H^*_2) - \frac{R(D,n)}{H^*_2}\right)\sum_{z=0}^{n-t}\beta^z + \beta^{n+1-t}\left(C + \frac{\pi(H^*_{n+2})}{1-\beta}\right) \\ R^*_t &= \pi(H^*_2)\sum_{z=0}^{n-t}\beta^z + \beta^{n+1-t}\frac{\pi(H^*_{n+2})}{1-\beta}. \end{aligned}$$

In period 0 the prices are

$$\begin{aligned} P^*_0 &= (1-H_0)\overline{\theta} - \frac{G-D}{H_0} + \left((1-H^*_2)\overline{\theta} + B(H^*_2) - \frac{R(D,n)}{H^*_2}\right)\sum_{z=1}^{n}\beta^z + \beta^{n+1}\left(C + \frac{\pi(H^*_{n+2})}{1-\beta}\right) \\ R^*_0 &= \pi(H_0) + \pi(H^*_2)\sum_{z=1}^{n}\beta^z + \beta^{n+1}\frac{\pi(H^*_{n+2})}{1-\beta}. \end{aligned}$$

To show that $\{H^*_{t+1}, \widetilde{P}_t, R^*_t\}_{t=0}^{\infty}$ is an equilibrium with n period development tax τ^* where the tax τ^* and price sequence $\{\widetilde{P}_t\}_{t=0}^{\infty}$ are as defined in the statement of the Proposition, we first need to verify that there is development in period 1. This requires that

$$C + \frac{\pi(H_0)}{1-\beta} < \frac{(1-H_0)\overline{\theta} + B(H_0) - \tau^* / \sum_{z=0}^{n-1} \beta^z}{1-\beta}.$$

This follows from the fact that

$$\tau^* / \sum_{z=0}^{n-1} \beta^z = D/H_0 \sum_{z=0}^{n-1} \beta^{z+1} = D/H_0 \sum_{z=1}^{n} \beta^z = \frac{R(D,n)}{H_0}.$$

In terms of prices, we require that in periods $t = 2, ..., n$

$$\widetilde{P}_t = \left((1-H^*_2)\overline{\theta} + B(H^*_2)\right) \sum_{z=0}^{n-t} \beta^z + \beta^{n+1-t}\left(C + \frac{\pi(H^*_{n+2})}{1-\beta}\right).$$

But we know that for $t = 2, ..., n$, we have that

$$\begin{aligned}
\widetilde{P}_t &= P^*_t + \frac{\tau^* H_0 \sum_{z=1}^{n+1-t} \beta^z}{H^*_2 \sum_{z=1}^{n} \beta^z} \\
&= \left((1-H^*_2)\overline{\theta} + B(H^*_2) - \frac{R(D,n)}{H^*_2}\right) \sum_{z=0}^{n-t} \beta^z + \beta^{n+1-t}\left(C + \frac{\pi(H^*_{n+2})}{1-\beta}\right) + \frac{\tau^* H_0 \sum_{z=1}^{n+1-t} \beta^z}{H^*_2 \sum_{z=1}^{n} \beta^z}.
\end{aligned}$$

Thus, we need to show that

$$\frac{\tau^* H_0 \sum_{z=1}^{n+1-t} \beta^z}{H^*_2 \sum_{z=1}^{n} \beta^z} = \sum_{z=0}^{n-t} \beta^z \frac{R(D,n)}{H^*_2},$$

which is true. For period 1, we require that

$$\widetilde{P}_1 = \left((1-H^*_2)\overline{\theta} + B(H^*_2)\right) \sum_{z=0}^{n-1} \beta^z + \tau^* \frac{H^*_2 - H_0}{H^*_2} + \beta^n \left(C + \frac{\pi(H^*_{n+2})}{1-\beta}\right).$$

We have that

$$\widetilde{P}_1 = P^*_1 + \tau^* = \left((1-H^*_2)\overline{\theta} + B(H^*_2) - \frac{R(D,n)}{H^*_2}\right) \sum_{z=0}^{n-1} \beta^z + \beta^n \left(C + \frac{\pi(H^*_{n+2})}{1-\beta}\right) + \tau^*.$$

Thus, we need that

$$\tau^* \frac{H^*_2 - H_0}{H^*_2} = \tau^* - \frac{R(D,n)}{H^*_2} \sum_{z=0}^{n-1} \beta^z.$$

The left hand side is equal to

$$\tau^* \frac{H^*_2 - H_0}{H^*_2} = \tau^* - \frac{H_0}{H^*_2}\tau^* = \tau^* - \frac{D}{\beta H^*_2}.$$

The right hand side is equal to

$$\tau^* - \frac{R(D,n)}{H_2^*}\sum_{z=0}^{n-1}\beta^z = \tau^* - \frac{D}{H_2^*}\frac{\sum_{z=0}^{n-1}\beta^z}{\sum_{z=1}^{n}\beta^z} = \tau^* - \frac{D}{\beta H_2^*}.$$

For period 0, we require that

$$\widetilde{P}_0 = (1-H_0)\overline{\theta} - \frac{G}{H_0} + \left((1-H_0)\overline{\theta} + B(H_0)\right)\sum_{z=1}^{n}\beta^z + \beta^{n+1}\left(C + \frac{\pi(H_{n+2}^*)}{1-\beta}\right).$$

We have

$$\widetilde{P}_0 = P_0^* = (1-H_0)\overline{\theta} - \frac{G-D}{H_0} + \left((1-H_0)\overline{\theta} + B(H_0) - \frac{R(D,n)}{H_0}\right)\sum_{z=1}^{n}\beta^z + \beta^{n+1}\left(C + \frac{\pi(H_{n+2}^*)}{1-\beta}\right).$$

Thus, we need to show that

$$\frac{D}{H_0} = \sum_{z=1}^{n}\beta^z\frac{R(D,n)}{H_0},$$

which is true.

It remains to show that agents have the same payoffs in the two equilibria. This is immediate for period $t \geq n+1$ potential residents and for landowners. We therefore focus on the period $t = 0, ..., n$ potential residents and the initial residents. Consider first the period t potential residents for $t \in \{2, ..., n\}$. In the equilibrium with debt $\{H_{t+1}^*, P_t^*, R_t^*\}_{t=0}^{\infty}$, if θ exceeds $(1-H_2^*)\overline{\theta}$, such a household gets a payoff

$$V_{\theta t}^* = \frac{\theta + (\mu\beta)^{n+1-t}B(H_{n+2}^*)}{1-\mu\beta} + \left(B(H_2^*) - \frac{R(D,n)}{H_2^*}\right)\sum_{z=0}^{n-t}(\mu\beta)^z - P_t^*$$
$$+(1-\mu)\beta\left[\sum_{z=1}^{n-t}(\mu\beta)^{z-1}P_{t+z}^* + \frac{(\mu\beta)^{n-t}P_{n+1}^*}{1-\mu\beta}\right].$$

If θ is between $(1-H_{n+2}^*)\overline{\theta}$ and $(1-H_2^*)\overline{\theta}$ a period t potential resident obtains a payoff $(\beta\mu)^{n+1-t}V_{\theta n+1}^*$. In the equilibrium with development tax τ^* $\{H_{t+1}^*, \widetilde{P}_t, R_t^*\}_{t=0}^{\infty}$, if θ exceeds $(1-H_2^*)\overline{\theta}$, such a household gets

$$\widetilde{V}_{\theta t} = \frac{\theta + (\mu\beta)^{n+1-t}B(H_{n+2}^*)}{1-\mu\beta} + B(H_2^*)\sum_{z=0}^{n-t}(\mu\beta)^z - \widetilde{P}_t + (1-\mu)\beta\left[\sum_{z=1}^{n-t}(\mu\beta)^{z-1}\widetilde{P}_{t+z} + \frac{(\mu\beta)^{n-t}\widetilde{P}_{n+1}}{1-\mu\beta}\right].$$

If θ is between $(1-H_{n+2}^*)\overline{\theta}$ and $(1-H_2^*)\overline{\theta}$, $\widetilde{V}_{\theta t}$ is equal to $\beta\mu\widetilde{V}_{\theta n+1}$.

Since $\widetilde{V}_{\theta n+1} = V_{\theta n+1}^*$, payoff equality holds for θ less than $(1-H_{n+2}^*)\overline{\theta}$. For θ greater than $(1-H_{n+2}^*)\overline{\theta}$ we need that

$$-\widetilde{P}_t + (1-\mu)\beta\left[\sum_{z=1}^{n-t}(\mu\beta)^{z-1}\widetilde{P}_{t+z} + \frac{(\mu\beta)^{n-t}\widetilde{P}_{n+1}}{1-\mu\beta}\right]$$
$$= -\frac{R(D,n)}{H_2^*}\sum_{z=0}^{n-t}(\mu\beta)^z - P_t^* + (1-\mu)\beta\left[\sum_{z=1}^{n-t}(\mu\beta)^{z-1}P_{t+z}^* + \frac{(\mu\beta)^{n-t}P_{n+1}^*}{1-\mu\beta}\right].$$

Since $\widetilde{P}_{n+1} = P^*_{n+1}$, this equality reduces to:

$$-\widetilde{P}_t + (1-\mu)\beta \sum_{z=1}^{n-t} (\mu\beta)^{z-1} \widetilde{P}_{t+z} = -\frac{R(D,n)}{H_2^*} \sum_{z=0}^{n-t} (\mu\beta)^z - P_t^* + (1-\mu)\beta \sum_{z=1}^{n-t} (\mu\beta)^{z-1} P^*_{t+z}.$$

We have that

$$\begin{gathered} -\widetilde{P}_t + (1-\mu)\beta \textstyle\sum_{z=1}^{n-t} (\mu\beta)^{z-1} \widetilde{P}_{t+z} \\ = -P_t^* - \frac{\tau^* H_0 \sum_{z=1}^{n+1-t} \beta^z}{H_2^* \sum_{z=1}^{n} \beta^z} + (1-\mu)\beta \textstyle\sum_{z=1}^{n-t} (\mu\beta)^{z-1} \left(P^*_{t+z} + \frac{\tau^* H_0 \sum_{w=1}^{n+1-t-z} \beta^w}{H_2^* \sum_{w=1}^{n} \beta^w} \right). \end{gathered}$$

We therefore require that

$$-\frac{\tau^* H_0 \sum_{z=1}^{n+1-t} \beta^z}{H_2^* \sum_{z=1}^{n} \beta^z} + (1-\mu)\beta \sum_{z=1}^{n-t} (\mu\beta)^{z-1} \left(\frac{\tau^* H_0 \sum_{w=1}^{n+1-t-z} \beta^w}{H_2^* \sum_{w=1}^{n} \beta^w} \right) = -\frac{R(D,n)}{H_2^*} \sum_{z=0}^{n-t} (\mu\beta)^z .$$

We have that

$$\begin{aligned} & -\frac{\tau^* H_0 \sum_{z=1}^{n+1-t} \beta^z}{H_2^* \sum_{z=1}^{n} \beta^z} + (1-\mu)\beta \sum_{z=1}^{n-t} (\mu\beta)^{z-1} \left(\frac{\tau^* H_0 \sum_{w=1}^{n+1-t-z} \beta^w}{H_2^* \sum_{w=1}^{n} \beta^w} \right) \\ = & -\frac{\tau^* H_0}{H_2^* \sum_{z=1}^{n} \beta^z} \left(\sum_{z=1}^{n+1-t} \beta^z - (1-\mu)\beta \sum_{z=1}^{n-t} (\mu\beta)^{z-1} \sum_{w=1}^{n+1-t-z} \beta^w \right) \\ = & -\frac{R(D,n)}{\beta H_2^*} \left(\sum_{z=1}^{n+1-t} \beta^z - (1-\mu)\beta \sum_{z=1}^{n-t} (\mu\beta)^{z-1} \sum_{w=1}^{n+1-t-z} \beta^w \right). \end{aligned}$$

We now show that

$$\sum_{z=1}^{n+1-t} \beta^z - (1-\mu)\beta \sum_{z=1}^{n-t} (\mu\beta)^{z-1} \sum_{w=1}^{n+1-t-z} \beta^w = \beta \left(\sum_{z=0}^{n-t} (\mu\beta)^z \right).$$

To prove this, note first that

$$\begin{aligned} & \sum_{z=1}^{n+1-t} \beta^z - (1-\mu)\beta \sum_{z=1}^{n-t} (\mu\beta)^{z-1} \sum_{w=1}^{n+1-t-z} \beta^w \\ = & \beta \left[\sum_{z=1}^{n+1-t} \beta^{z-1} - (1-\mu) \sum_{z=1}^{n-t} (\mu\beta)^{z-1} \sum_{w=1}^{n+1-t-z} \beta^w \right] \\ = & \beta \left[\sum_{z=0}^{n-t} \beta^z - \sum_{z=1}^{n-t} (\mu\beta)^{z-1} \sum_{w=1}^{n+1-t-z} \beta^w + \mu \sum_{z=1}^{n-t} (\mu\beta)^{z-1} \sum_{w=1}^{n+1-t-z} \beta^w \right]. \end{aligned}$$

In addition, we have that

$$\begin{aligned} & \sum_{z=1}^{n-t} (\mu\beta)^{z-1} \sum_{w=1}^{n+1-t-z} \beta^w \\ = & \sum_{w=1}^{n-t} \beta^w + (\mu\beta) \sum_{w=1}^{n-t-1} \beta^w + (\mu\beta)^2 \sum_{w=1}^{n-t-2} \beta^w + (\mu\beta)^3 \sum_{w=1}^{n-t-3} \beta^w + (\mu\beta)^4 \sum_{w=1}^{n-t-4} \beta^w .... + (\mu\beta)^{n-t-1} \beta \\ = & \sum_{w=1}^{n-t} \beta^w + \mu \sum_{w=2}^{n-t} \beta^w + \mu^2\beta \sum_{w=2}^{n-t-1} \beta^w + \mu^3\beta^2 \sum_{w=2}^{n-t-2} \beta^w + \mu^4\beta^3 \sum_{w=2}^{n-t-3} \beta^w + ... + \mu^{n-t-1}\beta^{n-t-2} \sum_{w=2}^{2} \beta^w \end{aligned}$$

and

$$
\begin{aligned}
& \mu \sum_{z=1}^{n-t} (\mu\beta)^{z-1} \sum_{w=1}^{n+1-t-z} \beta^w \\
= \quad & \mu \sum_{w=1}^{n-t} \beta^w + \mu^2\beta \sum_{w=1}^{n-t-1} \beta^w + \mu^3\beta^2 \sum_{w=1}^{n-t-2} \beta^w + \mu^4\beta^3 \sum_{w=1}^{n-t-3} \beta^w + ... + +\mu^{n-t-1}\beta^{n-t-2} \sum_{w=1}^{2} \beta^w + \mu^{n-t}\beta^{n-t-1}\beta.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& \sum_{z=0}^{n-t} \beta^z - \sum_{z=1}^{n-t} (\mu\beta)^{z-1} \sum_{w=1}^{n+1-t-z} \beta^w + \mu \sum_{z=1}^{n-t} (\mu\beta)^{z-1} \sum_{w=1}^{n+1-t-z} \beta^w \\
= \quad & \sum_{z=0}^{n-t} \beta^z - \left(\sum_{w=1}^{n-t} \beta^w + \mu \sum_{w=2}^{n-t} \beta^w + \mu^2\beta \sum_{w=2}^{n-t-1} \beta^w + \mu^3\beta^2 \sum_{w=2}^{n-t-2} \beta^w + \mu^4\beta^3 \sum_{w=2}^{n-t-3} \beta^w + ... + \mu^{n-t-1}\beta^{n-t-2} \sum_{w=2}^{2} \beta^w \right) \\
& + \left(\mu \sum_{w=1}^{n-t} \beta^w + \mu^2\beta \sum_{w=1}^{n-t-1} \beta^w + \mu^3\beta^2 \sum_{w=1}^{n-t-2} \beta^w + \mu^4\beta^3 \sum_{w=1}^{n-t-3} \beta^w + ... + \mu^{n-t-1}\beta^{n-t-2} \sum_{w=1}^{2} \beta^w + \mu^{n-t}\beta^{n-t-1}\beta \right) \\
= \quad & \sum_{z=0}^{n-t} \beta^z - \sum_{w=1}^{n-t} \beta^w + \mu \sum_{w=1}^{n-t} \beta^w - \mu \sum_{w=2}^{n-t} \beta^w + \mu^2\beta \sum_{w=1}^{n-t-1} \beta^w - \mu^2\beta \sum_{w=2}^{n-t-1} \beta^w + \mu^3\beta^2 \sum_{w=1}^{n-t-2} \beta^w - \mu^3\beta^2 \sum_{w=2}^{n-t-2} \beta^w \\
& +\mu^4\beta^3 \sum_{w=1}^{n-t-3} \beta^w - \mu^4\beta^3 \sum_{w=2}^{n-t-3} \beta^w + ... + \mu^{n-t-1}\beta^{n-t-2} \sum_{w=1}^{2} \beta^w - \mu^{n-t-1}\beta^{n-t-2} \sum_{w=2}^{2} \beta^w + \mu^{n-t}\beta^{n-t-1}\beta \\
= \quad & \sum_{z=0}^{n-t} (\mu\beta)^z
\end{aligned}
$$

as required.

Next consider the period 1 potential residents. In the equilibrium with debt $\{H^*_{t+1}, P^*_t, R^*_t\}_{t=0}^{\infty}$, if θ exceeds $(1 - H^*_2)\overline{\theta}$, such a household gets a payoff

$$
\begin{aligned}
V^*_{\theta 1} = & \tfrac{\theta + (\mu\beta)^n B(H^*_{n+2})}{1-\mu\beta} + \left(B(H^*_2) - \tfrac{R(D,n)}{H^*_2} \right) \textstyle\sum_{z=0}^{n-1} (\mu\beta)^z - P^*_1 \\
& +(1-\mu)\beta \left[\textstyle\sum_{z=1}^{n-1} (\mu\beta)^{z-1} P^*_{1+z} + \tfrac{(\mu\beta)^{n-1} P^*_{n+1}}{1-\mu\beta} \right].
\end{aligned}
$$

If θ is between $(1-H^*_{n+2})\overline{\theta}$ and $(1-H^*_2)\overline{\theta}$ a period 1 potential resident obtains a payoff $(\beta\mu)^n V^*_{\theta n+1}$. In the equilibrium with development tax τ^* $\{H^*_{t+1}, \widetilde{P}_t, R^*_t\}_{t=0}^{\infty}$, if θ exceeds $(1 - H^*_2)\overline{\theta}$, such a household gets

$$
\begin{aligned}
\widetilde{V}_{\theta 1} = & \tfrac{\theta + (\mu\beta)^n B(H^*_{n+2})}{1-\mu\beta} + \tau \tfrac{(H^*_2 - H_0)}{H^*_2} - \widetilde{P}_1 \\
+B(H^*_2) \textstyle\sum_{z=0}^{n-1} (\mu\beta)^z & + (1-\mu)\beta \left[\textstyle\sum_{z=1}^{n-1} (\mu\beta)^{z-1} \widetilde{P}_{1+z} + \tfrac{(\mu\beta)^{n-1} \widetilde{P}_{n+1}}{1-\mu\beta} \right].
\end{aligned}
$$

If θ is between $(1 - H^*_{n+2})\overline{\theta}$ and $(1 - H^*_2)\overline{\theta}$, $\widetilde{V}_{\theta t}$ is equal to $\beta\mu\widetilde{V}_{\theta n+1}$. Since $\widetilde{V}_{\theta n+1} = V^*_{\theta n+1}$, payoff equality holds for θ less than $(1-H^*_{n+2})\overline{\theta}$. For θ greater than $(1-H^*_{n+2})\overline{\theta}$ we need that

$$\tau^*\tfrac{(H_2^*-H_0)}{H_2^*}-\widetilde{P}_1+(1-\mu)\beta\left[\sum_{z=1}^{n-1}(\mu\beta)^{z-1}\widetilde{P}_{1+z}+\tfrac{(\mu\beta)^{n-1}\widetilde{P}_{n+1}}{1-\mu\beta}\right]$$
$$=-\tfrac{R(D,n)}{H_2^*}\sum_{z=0}^{n-1}(\mu\beta)^z-P_1^*+(1-\mu)\beta\left[\sum_{z=1}^{n-1}(\mu\beta)^{z-1}P^*_{1+z}+\tfrac{(\mu\beta)^{n-1}P^*_{n+1}}{1-\mu\beta}\right].$$

Since $\widetilde{P}_{n+1}=P^*_{n+1}$, this equality reduces to:

$$\tau^*\frac{(H_2^*-H_0)}{H_2^*}-\widetilde{P}_1+(1-\mu)\beta\sum_{z=1}^{n-1}(\mu\beta)^{z-1}\widetilde{P}_{1+z}=-\frac{R(D,n)}{H_2^*}\sum_{z=0}^{n-1}(\mu\beta)^z-P_1^*+(1-\mu)\beta\sum_{z=1}^{n-1}(\mu\beta)^{z-1}P^*_{1+z}.$$

We know that

$$\begin{aligned}&\tau^*\frac{(H_2^*-H_0)}{H_2^*}-\widetilde{P}_1+(1-\mu)\beta\sum_{z=1}^{n-1}(\mu\beta)^{z-1}\widetilde{P}_{1+z}\\ =\;&-\tau^*\frac{H_0}{H_2^*}-P_1^*+(1-\mu)\beta\sum_{z=1}^{n-1}(\mu\beta)^{z-1}\left(P^*_{1+z}+\frac{\tau^*H_0\sum_{w=1}^{n-z}\beta^w}{H_2^*\sum_{w=1}^{n}\beta^w}\right)\end{aligned}$$

We therefore require that

$$-\frac{\tau^*H_0\sum_{z=1}^{n}\beta^z}{H_2^*\sum_{z=1}^{n}\beta^z}+(1-\mu)\beta\sum_{z=1}^{n-1}(\mu\beta)^{z-1}\left(\frac{\tau^*H_0\sum_{w=1}^{n-z}\beta^w}{H_2^*\sum_{w=1}^{n}\beta^w}\right)=-\frac{R(D,n)}{H_2^*}\sum_{z=0}^{n-1}(\mu\beta)^z.$$

We have that

$$\begin{aligned}&-\frac{\tau^*H_0\sum_{z=1}^{n}\beta^z}{H_2^*\sum_{z=1}^{n}\beta^z}+(1-\mu)\beta\sum_{z=1}^{n-1}(\mu\beta)^{z-1}\left(\frac{\tau^*H_0\sum_{w=1}^{n-z}\beta^w}{H_2^*\sum_{w=1}^{n}\beta^w}\right)\\ =\;&-\frac{\tau^*H_0}{H_2^*\sum_{z=1}^{n}\beta^z}\left(\sum_{z=1}^{n}\beta^z-(1-\mu)\beta\sum_{z=1}^{n-1}(\mu\beta)^{z-1}\sum_{w=1}^{n-z}\beta^w\right)\\ =\;&-\frac{R(D,n)}{\beta H_2^*}\left(\sum_{z=1}^{n}\beta^z-(1-\mu)\beta\sum_{z=1}^{n-1}(\mu\beta)^{z-1}\sum_{w=1}^{n-z}\beta^w\right).\end{aligned}$$

As shown above, it is the case that

$$\sum_{z=1}^{n}\beta^z-(1-\mu)\beta\sum_{z=1}^{n-1}(\mu\beta)^{z-1}\sum_{w=1}^{n-z}\beta^w=\beta\left(\sum_{z=0}^{n-1}(\mu\beta)^z\right).$$

The desired equality therefore follows.

Next consider the period 0 potential residents. In the equilibrium with debt $\{H^*_{t+1},P^*_t,R^*_t\}_{t=0}^{\infty}$, if θ exceeds $(1-H_0)\overline{\theta}$, such a household gets a payoff

$$V^*_{\theta 0}=\tfrac{\theta+(\mu\beta)^{n+1}B(H^*_{n+2})}{1-\mu\beta}+\left(B(H_2^*)-\tfrac{R(D,n)}{H_2^*}\right)\sum_{z=1}^{n}(\mu\beta)^z-\tfrac{G-D}{H_0}-P_0^*$$
$$+(1-\mu)\beta\left[\sum_{z=0}^{n-1}(\mu\beta)^zP^*_{1+z}+\tfrac{(\mu\beta)^nP^*_{n+1}}{1-\mu\beta}\right].$$

If θ is between $(1-H_2^*)\overline{\theta}$ and $(1-H_0)\overline{\theta}$ a period 0 potential resident obtains a payoff equal to $\beta\mu V_{\theta 1}^*$. If θ is between $(1-H_{n+2}^*)\overline{\theta}$ and $(1-H_2^*)\overline{\theta}$ a period 0 potential resident obtains a payoff equal to $(\beta\mu)^{n+1} V_{\theta n+1}^*$. In the equilibrium with development tax τ^* $\{H_{t+1}^*, \widetilde{P}_t, R_t^*\}_{t=0}^{\infty}$, if θ exceeds $(1-H_0)\overline{\theta}$, a period 0 potential resident gets

$$\begin{aligned}\widetilde{V}_{\theta 0} = & \tfrac{\theta+(\mu\beta)^{n+1}B(H_{n+2}^*)}{1-\mu\beta} + B(H_2^*)\textstyle\sum_{z=1}^{n}(\mu\beta)^z + \beta\mu\tau^*\tfrac{(H_2^*-H_0)}{H_2^*} - \tfrac{G}{H_0} - \widetilde{P}_0 \\ & +(1-\mu)\beta\left[\textstyle\sum_{z=0}^{n-1}(\mu\beta)^z\widetilde{P}_{1+z} + \tfrac{(\mu\beta)^n\widetilde{P}_{n+1}}{1-\mu\beta}\right].\end{aligned}$$

If θ is between $(1-H_2^*)\overline{\theta}$ and $(1-H_0)\overline{\theta}$ a period 0 potential resident obtains a payoff equal to $\beta\mu\widetilde{V}_{\theta 1}$. If θ is between $(1-H_{n+2}^*)\overline{\theta}$ and $(1-H_2^*)\overline{\theta}$ a period 0 potential resident obtains a payoff equal to $(\beta\mu)^{n+1}\widetilde{V}_{\theta n+1}$.

Since $\widetilde{V}_{\theta 1} = V_{\theta 1}^*$ and $\widetilde{V}_{\theta n+1} = V_{\theta n+1}^*$, payoff equality holds for θ less than $(1-H_0^*)\overline{\theta}$. For θ greater than $(1-H_0^*)\overline{\theta}$ we need that

$$\begin{aligned} & \beta\mu\tau^*\frac{(H_2^*-H_0)}{H_2^*} - \widetilde{P}_0 + (1-\mu)\beta\left[\sum_{z=0}^{n-1}(\mu\beta)^z\widetilde{P}_{1+z} + \frac{(\mu\beta)^n\widetilde{P}_{n+1}}{1-\mu\beta}\right] \\ = & -\frac{R(D,n)}{H_2^*}\sum_{z=1}^{n}(\mu\beta)^z + \frac{D}{H_0} - P_0^* + (1-\mu)\beta\left[\sum_{z=0}^{n-1}(\mu\beta)^z P_{1+z}^* + \frac{(\mu\beta)^n P_{n+1}^*}{1-\mu\beta}\right].\end{aligned}$$

Since $\widetilde{P}_{n+1} = P_{n+1}^*$ and $\widetilde{P}_0 = P_0^*$, this equality reduces to:

$$\beta\mu\tau^*\frac{(H_2^*-H_0)}{H_2^*} + (1-\mu)\beta\sum_{z=0}^{n-1}(\mu\beta)^z\widetilde{P}_{1+z} = -\frac{R(D,n)}{H_2^*}\sum_{z=1}^{n}(\mu\beta)^z + \frac{D}{H_0} + (1-\mu)\beta\sum_{z=0}^{n-1}(\mu\beta)^z P_{1+z}^*.$$

We know that

$$\begin{aligned} & \beta\mu\tau^*\frac{(H_2^*-H_0)}{H_2^*} + (1-\mu)\beta\sum_{z=0}^{n-1}(\mu\beta)^z\widetilde{P}_{1+z} \\ = & \beta\mu\tau^*\frac{(H_2^*-H_0)}{H_2^*} + (1-\mu)\beta\left[P_1^* + \tau^* + \sum_{z=1}^{n-1}(\mu\beta)^{z-1}\left(P_{1+z}^* + \frac{\tau^* H_0\sum_{w=1}^{n-z}\beta^w}{H_2^*\sum_{w=1}^{n}\beta^w}\right)\right].\end{aligned}$$

Should be

$$\begin{aligned} & \beta\mu\tau^*\frac{(H_2^*-H_0)}{H_2^*} + (1-\mu)\beta\sum_{z=0}^{n-1}(\mu\beta)^z\widetilde{P}_{1+z} \\ = & \beta\mu\tau^*\frac{(H_2^*-H_0)}{H_2^*} + (1-\mu)\beta\left[P_1^* + \tau^* + \sum_{z=1}^{n-1}(\mu\beta)^{z}\left(P_{1+z}^* + \frac{\tau^* H_0\sum_{w=1}^{n-z}\beta^w}{H_2^*\sum_{w=1}^{n}\beta^w}\right)\right]\end{aligned}$$

Thus, we need to show that

$$\beta\mu\tau^*\frac{(H_2^*-H_0)}{H_2^*} + (1-\mu)\beta\left[\tau^* + \sum_{z=1}^{n-1}(\mu\beta)^z\left(\frac{\tau^* H_0\sum_{w=1}^{n-z}\beta^w}{H_2^*\sum_{w=1}^{n}\beta^w}\right)\right] = -\frac{R(D,n)}{H_2^*}\sum_{z=1}^{n}(\mu\beta)^z + \frac{D}{H_0}$$

or, equivalently, that

$$\beta\tau^* - \beta\mu\tau^*\frac{H_0}{H_2^*} + (1-\mu)\beta\sum_{z=1}^{n-1}(\mu\beta)^z\left(\frac{\tau^* H_0 \sum_{w=1}^{n-z}\beta^w}{H_2^*\sum_{w=1}^{n}\beta^w}\right) = -\frac{R(D,n)}{H_2^*}\sum_{z=1}^{n}(\mu\beta)^z + \frac{D}{H_0}.$$

Using the definition of τ^*, this becomes

$$-\beta\mu\tau^*\frac{H_0}{H_2^*} + (1-\mu)\beta\sum_{z=1}^{n-1}(\mu\beta)^z\left(\frac{\tau^* H_0 \sum_{w=1}^{n-z}\beta^w}{H_2^*\sum_{w=1}^{n}\beta^w}\right) = -\frac{R(D,n)}{H_2^*}\sum_{z=1}^{n}(\mu\beta)^z$$

$$\begin{aligned}
& -\beta\mu\frac{\tau^* H_0 \sum_{z=1}^{n}\beta^z}{H_2^*\sum_{z=1}^{n}\beta^z} + (1-\mu)\beta\sum_{z=1}^{n-1}(\mu\beta)^z\left(\frac{\tau^* H_0 \sum_{w=1}^{n-z}\beta^w}{H_2^*\sum_{w=1}^{n}\beta^w}\right) \\
= & -\frac{\tau^* H_0}{H_2^*\sum_{z=1}^{n}\beta^z}\left(\beta\mu\sum_{z=1}^{n}\beta^z - (1-\mu)\beta\sum_{z=1}^{n-1}(\mu\beta)^z\sum_{w=1}^{n-z}\beta^w\right) \\
= & -\frac{R(D,n)}{\beta H_2^*}\left(\beta\mu\sum_{z=1}^{n}\beta^z - (1-\mu)\beta\sum_{z=1}^{n-1}(\mu\beta)^z\sum_{w=1}^{n-z}\beta^w\right)
\end{aligned}$$

We need to show that

$$\beta\mu\sum_{z=1}^{n}\beta^z - (1-\mu)\beta\sum_{z=1}^{n-1}(\mu\beta)^z\sum_{w=1}^{n-z}\beta^w = \beta\sum_{z=1}^{n}(\mu\beta)^z$$

To prove this, note first that

$$\begin{aligned}
& \beta\mu\sum_{z=1}^{n}\beta^z - (1-\mu)\beta\sum_{z=1}^{n-1}(\mu\beta)^z\sum_{w=1}^{n-z}\beta^w \\
= & \beta\left[\beta\mu\sum_{z=1}^{n}\beta^{z-1} - (1-\mu)\sum_{z=1}^{n-1}(\mu\beta)^z\sum_{w=1}^{n-z}\beta^w\right] \\
= & \beta\left[\beta\mu\sum_{z=0}^{n-1}\beta^z - \sum_{z=1}^{n-1}(\mu\beta)^z\sum_{w=1}^{n-z}\beta^w + \mu\sum_{z=1}^{n-1}(\mu\beta)^z\sum_{w=1}^{n-z}\beta^w\right].
\end{aligned}$$

In addition, we have that

$$\begin{aligned}
& \sum_{z=1}^{n-1}(\mu\beta)^z\sum_{w=1}^{n-z}\beta^w \\
= & (\mu\beta)\sum_{w=1}^{n-1}\beta^w + (\mu\beta)^2\sum_{w=1}^{n-2}\beta^w + (\mu\beta)^3\sum_{w=1}^{n-3}\beta^w + (\mu\beta)^4\sum_{w=1}^{n-4}\beta^w + (\mu\beta)^5\sum_{w=1}^{n-5}\beta^w .... + (\mu\beta)^{n-1}\beta \\
= & (\mu\beta)\sum_{w=1}^{n-1}\beta^w + \mu^2\beta\sum_{w=2}^{n-1}\beta^w + \mu^3\beta^2\sum_{w=2}^{n-2}\beta^w + \mu^4\beta^3\sum_{w=2}^{n-3}\beta^w + \mu^5\beta^4\sum_{w=2}^{n-4}\beta^w + ... + \mu^{n-1}\beta^{n-2}\sum_{w=2}^{2}\beta^w
\end{aligned}$$

and

$$\begin{aligned}
& \mu\sum_{z=1}^{n-1}(\mu\beta)^z\sum_{w=1}^{n-z}\beta^w \\
= & \mu^2\beta\sum_{w=1}^{n-1}\beta^w + \mu^3\beta^2\sum_{w=1}^{n-2}\beta^w + \mu^4\beta^3\sum_{w=1}^{n-3}\beta^w + \mu^5\beta^4\sum_{w=1}^{n-4}\beta^w + ... + \mu^{n-1}\beta^{n-2}\sum_{w=1}^{2}\beta^w + \mu^n\beta^{n-1}\beta.
\end{aligned}$$

Thus, we have that

$$\mu \sum_{z=1}^{n-1} (\mu\beta)^z \sum_{w=1}^{n-z} \beta^w - \sum_{z=1}^{n-1} (\mu\beta)^z \sum_{w=1}^{n-z} \beta^w = -(\mu\beta) \sum_{w=1}^{n-1} \beta^w + (\mu\beta)^2 + ... + (\mu\beta)^n$$

It follows that

$$\begin{aligned} & \beta\mu \sum_{z=0}^{n-1} \beta^z - \sum_{z=1}^{n-1} (\mu\beta)^z \sum_{w=1}^{n-z} \beta^w + \mu \sum_{z=1}^{n-1} (\mu\beta)^z \sum_{w=1}^{n-z} \beta^w \\ = \ & \beta\mu \sum_{z=0}^{n-1} \beta^z - (\mu\beta) \sum_{w=1}^{n-1} \beta^w + (\mu\beta)^2 + ... + (\mu\beta)^n \\ = \ & \beta\mu + (\mu\beta)^2 + ... + (\mu\beta)^n = \sum_{z=1}^{n} (\mu\beta)^z \end{aligned}$$

as required.

Finally, consider the initial potential residents. In the equilibrium with debt, the initial residents get $V_\theta^* = \mu V_{\theta 0}^* + P_0^*$, while in the equilibrium with a development tax they get $\widetilde{V}_\theta = \mu \widetilde{V}_{\theta 0} + \widetilde{P}_0$. We know that $\widetilde{P}_0 = P_0^*$ and we have just shown that $\widetilde{V}_{\theta 0} = V_{\theta 0}^*$. Thus, these payoffs are the same. The same logic applies for the remaining initial potential residents.

Implications of Proposition 5

Ricardian Equivalence

Proposition 6 *Suppose that debt takes the form of bonds that have an n period maturity. Then, Ricardian Equivalence holds if and only if $S(H_0) \leq 0$.*

Proof of Proposition 6 Suppose first that $S(H_0) \leq 0$. By definition, Ricardian Equivalence holds when the payoffs of all agents in the equilibrium with debt D in the form of n period bonds do not depend on D. Consider a particular level of $D > 0$. By the Proposition there exists an equilibrium with n period development tax $\tau = D/\beta H_0$ in which the payoffs of all agents are the same as in the equilibrium with debt D. Note that in this equilibrium there will be no development since

$$\frac{(1 - H_0)\overline{\theta} + B(H_0)}{1 - \beta} \leq C + \frac{\pi(H_0)}{1 - \beta}.$$

Thus, the equilibrium will be exactly the same as that which would arise if the project were tax financed and there were no development tax: i.e., the equilibrium with n period development tax 0. This equilibrium is obviously the same as the equilibrium with debt 0. Accordingly, all agents' payoffs in the equilibrium with debt D are the same as those in the equilibrium with debt 0 and Ricardian Equivalence holds.

For the converse, suppose that $S(H_0) > 0$. Note that in this case, the equilibrium with n period development tax 0, would involve development taking place in period 1 and the housing level $H_2(0)$, satisfies $S(H_2) = 0$. Now consider a particular level of debt $D > 0$. By the Proposition there exists an equilibrium with n period development tax $\tau = D/\beta H_0$ in which the payoffs of all agents are the same as in the equilibrium with debt level D. In this equilibrium, development will take place in period $n+1$ and possibly in period 1. The housing level emerging in period $n+1$, H_{n+2}, will equal $H_2(0)$. The housing level emerging in period 1 will equal H_0 if $S(H_0) \leq \tau / \sum_{z=0}^{n-1} \beta^z$, or will equal $H_2(\tau)$ where

$$S(H_2) = \frac{\tau H_0}{\sum_{z=0}^{n-1} \beta^z H_2}$$

Since $S(H)$ is decreasing, $H_2(\tau) < H_2(0)$. This means that, while the housing level will eventually be the same as in the equilibrium with 0 development tax, development will be delayed. This will impact some agents payoffs and thus payoffs of agents in the equilibrium with n period development tax τ are not the same as those in the equilibrium with n period development tax 0. It follows that the payoffs of all agents in the equilibrium with debt level D are not the same as those in the equilibrium with debt level 0 and thus Ricardian Equivalence does not hold. ■

Optimal debt

Proposition 7 *Suppose that debt takes the form of bonds that have an n period maturity and that Ricardian Equivalence does not hold. Then, if $S(H_0) \leq -H_0 B'(H_0)$, no development is optimal and any debt level at least as big as $\min\{G, H_0 S(H_0) \sum_{t=1}^{n} \beta^t\}$ is optimal. If $S(H_0) > -H_0 B'(H_0)$, the optimal amount of development is $H^o - H_0$ and the optimal debt level is $\min\{G, -(H^o)^2 B'(H^o) \sum_{t=1}^{n} \beta^t\}$.*

Proof of Proposition 7 Welfare in an equilibrium with debt D in the form of n period bonds, can be written as

$$\begin{aligned} W = \beta C H_0 - G + \int_{\theta(1-H_0)}^{\overline{\theta}} \theta \frac{d\theta}{\overline{\theta}} + L \int_{\pi(H_0)}^{\overline{\pi}} \pi \frac{d\pi}{\overline{\pi}-\underline{\pi}} \\ + \sum_{t=1}^{n} \beta^t \left(\int_{\theta(1-H_2)}^{\overline{\theta}} (\theta + B(H_2)) \frac{d\theta}{\overline{\theta}} + L \int_{\pi(H_2)}^{\overline{\pi}} \pi \frac{d\pi}{\overline{\pi}-\underline{\pi}} - \frac{\beta}{\sum_{t=1}^{n} \beta^t} C H_2 \right) + \beta^{n+1} C H_2 \\ + \frac{\beta^{n+1}}{1-\beta} \left(\int_{\theta(1-H_{n+2})}^{\overline{\theta}} (\theta + B(H_{n+2})) \frac{d\theta}{\overline{\theta}} + L \int_{\pi(H_{n+2})}^{\overline{\pi}} \pi \frac{d\pi}{\overline{\pi}-\underline{\pi}} - C(1-\beta) H_{n+2} \right). \end{aligned} \quad \text{(A19)}$$

Here H_2 is the size of the community in period 1 and H_{n+2} is the long run size of the community which is reached in period $n+1$. The planner cannot control H_{n+2} with debt, because all debt is repaid by then. Thus, all the planner can influence with debt is H_2.

Observe that the first derivative of welfare with respect to H_2 is

$$\sum_{t=1}^{n} \beta^t \left[(1-H_2)\overline{\theta} + B(H_2) + H_2 B'(H_2) - \pi(H_2) - \frac{\beta}{\sum_{t=1}^{n} \beta^t} C\right] + \beta^{n+1} C,$$

while the second derivative is

$$\sum_{t=1}^{n} \beta^t \left[-\overline{\theta} + 2B'(H_2) + H_2 B''(H_2) - \pi'(H_2)\right] < 0.$$

It follows that welfare is concave in H_2. Accordingly, if

$$(1-H_0)\overline{\theta} + B(H_0) + H_0 B'(H_0) \leq C(\frac{\beta}{\sum_{t=1}^{n} \beta^t} - \frac{\beta^{n+1}}{\sum_{t=1}^{n} \beta^t}) + \pi(H_0), \tag{A20}$$

the optimal level of H_2 from the planner's perspective is H_0 and no development in period 1 is optimal. Otherwise, some development in period 1 is optimal and the optimal period 1 community size satisfies the first order condition

$$(1-H_2)\overline{\theta} + B(H_2) + H_2 B'(H_2) = C(\frac{\beta}{\sum_{t=1}^{n} \beta^t} - \frac{\beta^{n+1}}{\sum_{t=1}^{n} \beta^t}) + \pi(H_2). \tag{A21}$$

Note that (A20) is equivalent to

$$S(H_0) \leq -H_0 B'(H_0) + C(\frac{\beta}{\sum_{t=1}^{n} \beta^t} - \frac{\beta^{n+1}}{\sum_{t=1}^{n} \beta^t}) - C(1-\beta)$$

which is equivalent to

$$S(H_0) \leq -H_0 B'(H_0).$$

Similarly, rewriting (A21), the optimal period 1 comunity size satisfies

$$S(H) = -H B'(H), \tag{A22}$$

and thus is equal to H^o as defined in Proposition 3.

Suppose first that no development in period 1 is optimal. In an equilibrium with debt D in the form of n period bonds, there will be no development in period 1 if

$$\frac{(1-H_0)\overline{\theta} + B(H_0) - \frac{R(D,n)}{H_0}}{1-\beta} \leq C + \frac{\pi(H_0)}{1-\beta}.$$

Thus, any debt level such that

$$\frac{R(D,n)}{H_0} \geq (1-H_0)\overline{\theta} + B(H_0) - C(1-\beta) - \pi(H_0),$$

will result in no development. This condition is equivalent to

$$D \geq \sum_{t=1}^{n} \beta^t H_0 S(H_0)$$

If

$$G < \sum_{t=1}^{n} \beta^t H_0 S(H_0),$$

then the planner cannot eliminate period 1 development through debt financing. Nonetheless, the best strategy in this case will be to fully debt finance, since this will minimize the amount of development. The optimal debt level in this case is therefore any level at least as big as

$$\min\{G, \sum_{t=1}^{n} \beta^t H_0 S(H_0)\}.$$

Now suppose that some development in period 1 is optimal. In an equilibrium with debt D in the form of n period bonds, if there is development, the level will satisfy

$$\frac{(1-H_2)\overline{\theta} + B(H_2) - \frac{R(D,n)}{H_2}}{1-\beta} = C + \frac{\pi(H_2)}{1-\beta},$$

or equivalently

$$(1-H_2)\overline{\theta} + B(H_2) - \pi(H_2) = C(1-\beta) + \frac{R(D,n)}{H_2}.$$

Thus, the optimal debt level is such that

$$(1-H^o)\overline{\theta} + B(H^o) - \pi(H^o) = C(1-\beta) + \frac{R(D,n)}{H^o}.$$

This implies that

$$\frac{R(D,n)}{H^o} = S(H^o),$$

which means that

$$D = \sum_{t=1}^{n} \beta^t H^o S(H^o).$$

Given (A22), this is equivalent to

$$D = -\left(H^o\right)^2 B'(H^o) \sum_{t=1}^{n} \beta^t.$$

If

$$G < -\left(H^o\right)^2 B'(H^o) \sum_{t=1}^{n} \beta^t,$$

then the planner cannot achieve the optimal level of period 1 development through debt financing. Again, the best strategy in this case will be to fully debt finance since this will minimize the amount of excess development. The optimal debt level in this case is therefore equal to $\min\{G, -\left(H^o\right)^2 B'(H^o) \sum_{t=1}^{n} \beta^t\}$. ■

Equilibrium debt

Proposition 8 *Suppose that debt takes the form of bonds that have an n period maturity and that Ricardian Equivalence does not hold. Then, if*

$$S(H_0) \leq -H_0 B'(H_0) + \left(\frac{\sum_{t=1}^{n} \beta^z - \mu \sum_{t=1}^{n} (\mu\beta)^t}{\sum_{z=1}^{n} \beta^t} \right) \overline{\theta} H_0, \tag{A23}$$

the initial residents prefer no development and any debt level at least as big as $\min\{G, \sum_{z=1}^{n} \beta^t H_0 S(H_0)\}$ *is an equilibrium debt level. If condition (A23) does not hold, the initial residents prefer a level of development* $H_2^e - H_0$*, where* H_2^e *satisfies*

$$S(H_2^e) = -H_2^e B'(H_2^e) + \left(\overline{\theta} + \pi'(H_2^e)\right)(H_2^e - H_0) + \left(\frac{\sum_{t=1}^{n} \beta^t - \mu \sum_{t=1}^{n} (\mu\beta)^t}{\sum_{t=1}^{n} \beta^t} \right) \overline{\theta} H_0, \tag{A24}$$

and the equilibrium debt level is $\min\{G, \sum_{z=1}^{n} \beta^t H_2^e S(H_2^e)\}$*. The equilibrium debt level is at least as high as the optimal level and strictly higher when the optimal debt level equals* $-\left(H^o\right)^2 B'(H^o) \sum_{z=1}^{n} \beta^t$*.*

Proof of Proposition 8 We solve for the initial residents' preferred development tax and then use Proposition 5 to infer the equilibrium debt level. Given that $V_\theta = \mu V_{\theta 0} + P_0$ and the expression for $V_{\theta 0}$ in (A15), the initial residents' optimal taxation problem can be posed as choosing a development tax τ to maximize the objective function

$$\max_{\tau} \mu \left(\begin{array}{c} \frac{\theta + (\mu\beta)^{n+1} B(H_{n+2})}{1-\mu\beta} + B(H_2) \sum_{z=1}^{n} (\mu\beta)^z + \beta\mu\tau \frac{(H_2 - H_0)}{H_2} - \frac{G}{H_0} - P_0 \\ +(1-\mu)\beta \left[\sum_{z=0}^{n-1} (\mu\beta)^z P_{1+z} + \frac{(\mu\beta)^n P_{n+1}}{1-\mu\beta} \right] \end{array} \right) + P_0, \tag{A25}$$

where the housing levels H_2 and H_{n+2} and the prices P_0, $\{P_t\}_{t=1}^{n}$, and P_{n+1} are those arising in the equilibrium with n period development tax τ. We know that H_{n+2} and P_{n+1} are independent of the tax τ and can thus be treated as constants.

Let

$$\frac{(1-H_0)\overline{\theta}+B(H_0)}{1-\beta}=C+\frac{\pi(H_0)}{1-\beta}+\frac{\widehat{\tau}}{\sum_{z=0}^{n-1}\beta^z(1-\beta)}.$$

From (A9), there will be no development if $\tau \geq \widehat{\tau}$. All tax rates higher than $\widehat{\tau}$ will yield the same payoff as $\widehat{\tau}$ since there will be no development and no revenues raised. Thus, these tax rates can be ignored.

If $\tau < \widehat{\tau}$, there will be development. From (A11), the level of development will be such that

$$(1-H_2)\overline{\theta}+B(H_2)=C(1-\beta)+\pi(H_2)+\frac{\tau H_0}{\sum_{z=0}^{n-1}\beta^z H_2}..$$

This implies that

$$\tau=\sum_{z=0}^{n-1}\beta^z\left(\frac{H_2\left[(1-H_2)\overline{\theta}+B(H_2)-C(1-\beta)-\pi(H_2)\right]}{H_0}\right). \tag{A26}$$

Note that

$$\widehat{\tau}=\sum_{z=0}^{n-1}\beta^z\left((1-H_0)\overline{\theta}+B(H_0)-C(1-\beta)-\pi(H_0)\right), \tag{A27}$$

so equation (A26) also holds for $\tau=\widehat{\tau}$. From the analysis in Section 10.7.1, the price of housing in periods $n+1$ and beyond is

$$P_t=\frac{(1-H_2)\overline{\theta}+B(H_2)}{1-\beta}.$$

As discussed in Section 10.7.1, the price of housing in periods $t=2,...,n$, is

$$P_t=\left((1-H_2)\overline{\theta}+B(H_2)\right)\sum_{z=0}^{n-t}\beta^z+\beta^{n+1-t}\left(C+\frac{\pi(H_{n+2})}{1-\beta}\right),$$

the price of housing in period 1 is

$$P_1=C+\pi(H_2)\sum_{z=0}^{n-1}\beta^z+\beta^n\frac{\pi(H_{n+2})}{1-\beta}+\tau=\left((1-H_2)\overline{\theta}+B(H_2)\right)\sum_{z=0}^{n-1}\beta^z+\frac{\tau(H_2-H_0)}{H_2}+\beta^n\left(C+\frac{\pi(H_{n+2})}{1-\beta}\right).$$

and the price of housing in period 0 is

$$\begin{aligned}P_0 &= (1-H_0)\overline{\theta}-\frac{G}{H_0}+\beta\left(C+\pi(H_2)\sum_{z=0}^{n-1}\beta^z+\beta^n\frac{\pi(H_{n+2})}{1-\beta}+\tau\right)\\ &= (1-H_0)\overline{\theta}-\frac{G}{H_0}+\beta P_1.\end{aligned}$$

We now substitute these into the objective function in problem (A25). Ignoring constants and variables not influenced by τ, this can be written as

$$\mu\left(B(H_2)\sum_{z=1}^{n}(\mu\beta)^z+\beta\mu\tau\frac{(H_2-H_0)}{H_2}-P_0+(1-\mu)\beta\left[\sum_{z=0}^{n-1}(\mu\beta)^z P_{1+z}\right]\right)+P_0. \tag{A28}$$

Note first that

$$\begin{aligned}\sum_{z=0}^{n-1}(\mu\beta)^z P_{1+z} &= P_1+(\mu\beta)P_2+......+(\mu\beta)^{n-1}P_n\\ &= \left((1-H_2)\overline{\theta}+B(H_2)\right)\sum_{z=0}^{n-1}\beta^z+\frac{\tau(H_2-H_0)}{H_2}+\beta^n\left(C+\frac{\pi(H_{n+2})}{1-\beta}\right)\\ &\quad+(\mu\beta)\left(\left((1-H_2)\overline{\theta}+B(H_2)\right)\sum_{z=0}^{n-2}\beta^z+\beta^{n-1}\left(C+\frac{\pi(H_{n+2})}{1-\beta}\right)\right)\\ &\quad+(\mu\beta)^2\left(\left((1-H_2)\overline{\theta}+B(H_2)\right)\sum_{z=0}^{n-3}\beta^z+\beta^{n-2}\left(C+\frac{\pi(H_{n+2})}{1-\beta}\right)\right)\\ &\quad+(\mu\beta)^{n-1}\left(\left((1-H_2)\overline{\theta}+B(H_2)\right)+\beta\left(C+\frac{\pi(H_{n+2})}{1-\beta}\right)\right)\end{aligned}$$

Thus,

$$\begin{aligned}&\sum_{z=0}^{n-1}(\mu\beta)^z P_{1+z}\\ =\;&\frac{\tau(H_2-H_0)}{H_2}+\left((1-H_2)\overline{\theta}+B(H_2)\right)\left[\sum_{z=0}^{n-1}\beta^z+(\mu\beta)\sum_{z=0}^{n-2}\beta^z+(\mu\beta)^2\sum_{z=0}^{n-3}\beta^z+....+(\mu\beta)^{n-1}\sum_{z=0}^{n-n}\beta^z\right]\\ &+\left(C+\frac{\pi(H_{n+2})}{1-\beta}\right)\left[\beta^n+(\mu\beta)\beta^{n-1}+(\mu\beta)^2\beta^{n-2}+....+(\mu\beta)^{n-1}\beta\right]\end{aligned}$$

This means that

$$\begin{aligned}&\mu(1-\mu)\beta\sum_{z=0}^{n-1}(\mu\beta)^z P_{1+z}\\ =\;&\mu(1-\mu)\beta\frac{\tau(H_2-H_0)}{H_2}\\ &+\mu(1-\mu)\beta\left((1-H_2)\overline{\theta}+B(H_2)\right)\left[\sum_{z=0}^{n-1}\beta^z+(\mu\beta)\sum_{z=0}^{n-2}\beta^z+(\mu\beta)^2\sum_{z=0}^{n-3}\beta^z+....+(\mu\beta)^{n-1}\sum_{z=0}^{n-n}\beta^z\right]\\ &+\mu(1-\mu)\beta\left(C+\frac{\pi(H_{n+2})}{1-\beta}\right)\left[\beta^n+(\mu\beta)\beta^{n-1}+(\mu\beta)^2\beta^{n-2}+....+(\mu\beta)^{n-1}\beta\right]\\ =\;&\mu(1-\mu)\beta\frac{\tau(H_2-H_0)}{H_2}+\mu(1-\mu)\beta\left((1-H_2)\overline{\theta}+B(H_2)\right)\sum_{t=1}^{n}\left[(\mu\beta)^{t-1}\sum_{z=0}^{n-t}\beta^z\right]\\ &+\mu(1-\mu)\beta\left(C+\frac{\pi(H_{n+2})}{1-\beta}\right)\sum_{t=0}^{n-1}(\mu\beta)^t\beta^{n-t}.\end{aligned}$$

Furthermore, we have that

$$\begin{aligned}
& P_0(1-\mu) \\
= \quad & (1-\mu)\left((1-H_0)\overline{\theta} - \frac{G}{H_0}\right) \\
& +(1-\mu)\beta\left(\left((1-H_2)\overline{\theta}+B(H_2)\right)\sum_{z=0}^{n-1}\beta^z + \frac{\tau(H_2-H_0)}{H_2} + \beta^n\left(C+\frac{\pi(H_{n+2})}{1-\beta}\right)\right) \\
= \quad & (1-\mu)\left((1-H_0)\overline{\theta} - \frac{G}{H_0}\right) + (1-\mu)\beta\frac{\tau(H_2-H_0)}{H_2} \\
& +(1-\mu)\beta\left((1-H_2)\overline{\theta}+B(H_2)\right)\sum_{z=0}^{n-1}\beta^z + (1-\mu)\beta\beta^n\left(C+\frac{\pi(H_{n+2})}{1-\beta}\right).
\end{aligned}$$

Thus, objective function (A28) is

$$\begin{aligned}
& \mu\left(B(H_2)\sum_{z=1}^{n}(\mu\beta)^z + \beta\mu\tau\frac{(H_2-H_0)}{H_2} - P_0 + (1-\mu)\beta\left[\sum_{z=0}^{n-1}(\mu\beta)^z P_{1+z}\right]\right) + P_0 \\
= \quad & \mu\left(B(H_2)\sum_{z=1}^{n}(\mu\beta)^z + \beta\mu\tau\frac{(H_2-H_0)}{H_2}\right) \\
& +(1-\mu)\left((1-H_0)\overline{\theta} - \frac{G}{H_0}\right) + (1-\mu)\beta\frac{\tau(H_2-H_0)}{H_2} \\
& +(1-\mu)\beta\left((1-H_2)\overline{\theta}+B(H_2)\right)\sum_{z=0}^{n-1}\beta^z + (1-\mu)\beta\beta^n\left(C+\frac{\pi(H_{n+2})}{1-\beta}\right) \\
& +\mu(1-\mu)\beta\frac{\tau(H_2-H_0)}{H_2} + \mu(1-\mu)\beta\left((1-H_2)\overline{\theta}+B(H_2)\right)\sum_{t=1}^{n}\left[(\mu\beta)^{t-1}\sum_{z=0}^{n-t}\beta^z\right] \\
& +\mu(1-\mu)\beta\left(C+\frac{\pi(H_{n+2})}{1-\beta}\right)\sum_{t=0}^{n-1}(\mu\beta)^t\beta^{n-t}
\end{aligned}$$

Ignoring constants, this reduces to

$$\begin{aligned}
& \mu\left(B(H_2)\sum_{z=1}^{n}(\mu\beta)^z + \beta\mu\frac{\tau(H_2-H_0)}{H_2}\right) + (1-\mu)\beta\frac{\tau(H_2-H_0)}{H_2} \\
& +(1-\mu)\beta\left((1-H_2)\overline{\theta}+B(H_2)\right)\sum_{z=0}^{n-1}\beta^z + \mu(1-\mu)\beta\frac{\tau(H_2-H_0)}{H_2} \\
& +\mu(1-\mu)\beta\left((1-H_2)\overline{\theta}+B(H_2)\right)\sum_{t=1}^{n}\left[(\mu\beta)^{t-1}\sum_{z=0}^{n-t}\beta^z\right]
\end{aligned}$$

Collecting terms, this equals

$$\begin{array}{c}\frac{\tau(H_2-H_0)}{H_2}\left[\beta\mu^2+(1-\mu)\beta+\mu(1-\mu)\beta\right]\\ +B(H_2)\left(\mu\sum_{z=1}^{n}(\mu\beta)^z+(1-\mu)\beta\sum_{z=0}^{n-1}\beta^z+\mu(1-\mu)\beta\sum_{t=1}^{n}\left[(\mu\beta)^{t-1}\sum_{z=0}^{n-t}\beta^z\right]\right)\\ +(1-H_2)\overline{\theta}\left((1-\mu)\beta\sum_{z=0}^{n-1}\beta^z+\mu(1-\mu)\beta\sum_{t=1}^{n}\left[(\mu\beta)^{t-1}\sum_{z=0}^{n-t}\beta^z\right]\right)\end{array} \tag{A29}$$

Note that

$$\left[\beta\mu^2+(1-\mu)\beta+\mu(1-\mu)\beta\right]=\beta$$

and

$$\begin{aligned}&\mu\sum_{z=1}^{n}(\mu\beta)^z+(1-\mu)\beta\sum_{z=0}^{n-1}\beta^z+\mu(1-\mu)\beta\sum_{t=1}^{n}\left[(\mu\beta)^{t-1}\sum_{z=0}^{n-t}\beta^z\right]\\ =&\sum_{z=1}^{n}\beta^z+\mu\sum_{z=1}^{n}(\mu\beta)^z-\mu\sum_{z=1}^{n}\beta^z+\sum_{t=1}^{n}\left[(\mu\beta)^t\sum_{z=0}^{n-t}\beta^z-\mu(\mu\beta)^t\sum_{z=0}^{n-t}\beta^z\right]\\ =&\sum_{z=1}^{n}\beta^z.\end{aligned}$$

The last line follows from the fact that

$$\mu\sum_{z=1}^{n}(\mu\beta)^z-\mu\sum_{z=1}^{n}\beta^z+\sum_{t=1}^{n}\left[(\mu\beta)^t\sum_{z=0}^{n-t}\beta^z-\mu(\mu\beta)^t\sum_{z=0}^{n-t}\beta^z\right]=0.$$

Furthermore, since

$$\mu\sum_{z=1}^{n}(\mu\beta)^z+(1-\mu)\beta\sum_{z=0}^{n-1}\beta^z+\mu(1-\mu)\beta\sum_{t=1}^{n}\left[(\mu\beta)^{t-1}\sum_{z=0}^{n-t}\beta^z\right]=\sum_{z=1}^{n}\beta^z,$$

then

$$\begin{aligned}(1-\mu)\beta\sum_{z=0}^{n-1}\beta^z+\mu(1-\mu)\beta\sum_{t=1}^{n}\left[(\mu\beta)^{t-1}\sum_{z=0}^{n-t}\beta^z\right]&=\sum_{z=1}^{n}\beta^z-\mu\sum_{z=1}^{n}(\mu\beta)^z\\ &=\sum_{z=1}^{n}\beta^z\left(1-\mu^{z+1}\right).\end{aligned}$$

Thus objective function (A29) is

$$\beta\frac{\tau(H_2-H_0)}{H_2}+\sum_{z=1}^{n}\beta^zB(H_2)+\left(\sum_{z=1}^{n}\beta^z-\mu\sum_{z=1}^{n}(\mu\beta)^z\right)(1-H_2)\overline{\theta}.$$

Accordingly, we can pose the initial residents' optimal tax problem as:

$$\max_{(\tau,H_2)}\left\{\begin{array}{c}\beta\frac{\tau(H_2-H_0)}{H_2}+\sum_{z=1}^{n}\beta^z B(H_2)+\left(\sum_{z=1}^{n}\beta^z-\mu\sum_{z=1}^{n}(\mu\beta)^z\right)(1-H_2)\overline{\theta}\\ \tau=\sum_{z=0}^{n-1}\beta^z\left(\frac{H_2\left[(1-H_2)\overline{\theta}+B(H_2)-C(1-\beta)-\pi(H_2)\right]}{H_0}\right)\\ H_2\geq H_0\end{array}\right\}. \tag{A30}$$

The constraints embodies the requirement that H_2 be an equilibrium housing level given τ. If the solution involves $H_2 = H_0$, then it involves no development and a tax of $\widehat{\tau}$. If $H_2 > H_0$, then the solution involves development and a tax less than $\widehat{\tau}$.

To solve problem (43), we substitute the expression for the tax into the objective function to produce a problem involving only the choice of H_2. After the substitution, the objective function is (ignoring constants)

$$\sum_{z=1}^{n}\beta^z\left(\frac{\left[(1-H_2)\overline{\theta}+B(H_2)-C(1-\beta)-\pi(H_2)\right](H_2-H_0)}{H_0}\right)$$
$$+\sum_{z=1}^{n}\beta^z B(H_2)+\left(\sum_{z=1}^{n}\beta^z-\mu\sum_{z=1}^{n}(\mu\beta)^z\right)(1-H_2)\overline{\theta}.$$

The first derivative of this objective function is

$$\sum_{z=1}^{n}\beta^z\left(\frac{(1-H_2)\overline{\theta}+B(H_2)-C(1-\beta)-\pi(H_2)}{H_0}\right)+\sum_{z=1}^{n}\beta^z\frac{\left[-\overline{\theta}+B'(H_2)-\pi'(H_2)\right](H_2-H_0)}{H_0}$$
$$+\sum_{z=1}^{n}\beta^z B'(H_2)-\left(\sum_{z=1}^{n}\beta^z-\mu\sum_{z=1}^{n}(\mu\beta)^z\right)\overline{\theta},$$

and the second derivative is

$$\sum_{z=1}^{n}\beta^z\frac{2\left[-\overline{\theta}+B'(H_2)-\pi'(H_2)\right]}{H_0}+\sum_{z=1}^{n}\beta^z\frac{B''(H_2)(H_2-H_0)}{H_0}+\sum_{z=1}^{n}\beta^z B''(H_2)<0.$$

The objective function is therefore strictly concave.

If the first derivative of the objective function is negative at H_0, the initial residents will want to prevent development. This condition boils down to

$$(1-H_0)\overline{\theta}+B(H_0)-C(1-\beta)-\pi(H_0)\leq -H_0B'(H_0)+\left(\frac{\sum_{z=1}^{n}\beta^z-\mu\sum_{z=1}^{n}(\mu\beta)^z}{\sum_{z=1}^{n}\beta^z}\right)\overline{\theta}H_0$$

which is (A23). Any tax at least as big as $\widehat{\tau}$ will choke off development. Given (A2), the corresponding debt level is therefore anything bigger than

$$\frac{R(D,n)}{H_0} = (1 - H_0)\overline{\theta} + B(H_0) - C(1-\beta) - \pi(H_0),$$

or, equivalently, bigger than $D = \sum_{t=1}^{n} \beta^t H_0 S(H_0)$. If $\sum_{t=1}^{n} \beta^t H_0 S(H_0)$ exceeds G, then the initial residents cannot completely choke off development with debt financing. Their best strategy is nonetheless to choose the highest level of debt possible. We conclude that the equilibrium level of debt if (A23) holds is any debt level at least as big as $\min\{G, \sum_{t=1}^{n} \beta^t H_0 S(H_0)\}$.

If (A23) does not hold, the initial residents will want the level of development that equates the first derivative of the objective function to zero. This housing level satisfies

$$(1 - H_2)\overline{\theta} + B(H_2) - C(1-\beta) - \pi(H_2)$$
$$= -H_2 B'(H_2) + \left[\overline{\theta} + \pi'(H_2)\right](H_2 - H_0) + \left(\frac{\sum_{z=1}^{n} \beta^z - \mu \sum_{z=1}^{n} (\mu\beta)^z}{\sum_{z=1}^{n} \beta^z} \right) \overline{\theta} H_0.$$

This housing level corresponds to H_2^e as defined in (A24). The associated tax is

$$\tau = \sum_{z=0}^{n-1} \beta^z \left(\frac{H_2^e \left[(1 - H_2^e)\overline{\theta} + B(H_2^e) - C(1-\beta) - \pi(H_2^e)\right]}{H_0} \right).$$

The corresponding debt level therefore satisfies

$$\frac{D}{\beta H_0} = \sum_{z=0}^{n-1} \beta^z \left(\frac{H_2^e \left[(1 - H_2^e)\overline{\theta} + B(H_2^e) - C(1-\beta) - \pi(H_2^e)\right]}{H_0} \right),$$

or, equivalently, $D = \sum_{t=1}^{n} \beta^t H_2^e S(H_2^e)$. If $\sum_{t=1}^{n} \beta^t H_2^e S(H_2^e)$ exceeds G, then the initial residents cannot completely choke off development with debt financing. Their best strategy is nonetheless to choose the highest level of debt possible. We conclude that the equilibrium level of debt if (A23) does not hold is $\min\{G, \sum_{t=1}^{n} \beta^t H_2^e S(H_2^e)\}$.

It remains to establish the claim that the equilibrium debt level is at least as high as the optimal level and strictly higher when the optimal debt level equals $-(H^o)^2 B'(H^o) \sum_{t=1}^{n} \beta^t$. Proposition 7 tells us that if $S(H_0) \leq -H_0 B'(H_0)$, any debt level at least as big as $\min\{G, \sum_{t=1}^{n} \beta^t H_0 S(H_0)\}$ is optimal, while if $S(H_0) > -H_0 B'(H_0)$, the optimal debt level is $\min\{G, -(H^o)^2 B'(H^o) \sum_{t=1}^{n} \beta^t\}$

where H^o satisfies $S(H^o) = -H^o B'(H^o)$. Note that since $S(H^o) = -H^o B'(H^o)$, we have that

$$\min\{G, -(H^o)^2 B'(H^o) \sum_{t=1}^{n} \beta^t\} = \min\{G, \sum_{t=1}^{n} \beta^t H^o S(H^o)\}.$$

To prove the claim, suppose first that (A23) does not hold. Then we know that $S(H_0) > -H_0 B'(H_0)$. Thus, we need to show that $H_2^e S(H_2^e) > H^o S(H^o)$. It is clear that $H_2^e < H^o$. Thus, it suffices to show that $HS(H)$ is decreasing for $H \in [H_2^e, H^o]$. Observe that

$$\frac{dHS(H)}{dH} = (1-H)\overline{\theta} + B(H) + HB'(H) - C(1-\beta) - \pi(H) - H\left(\overline{\theta} + \pi'(H)\right),$$

and

$$\frac{d^2 HS(H)}{dH^2} = 2B'(H) + HB''(H) - 2\left(\overline{\theta} + \pi'(H)\right) < 0.$$

Thus, $HS(H)$ is concave. Now note that

$$\begin{aligned} \frac{dH_2^e S(H_2^e)}{dH} &= (1-H_2^e)\overline{\theta} + B(H_2^e) + H_2^e B'(H_2^e) - C(1-\beta) - \pi(H_2^e) - \left(\overline{\theta} + \pi'(H_2^e)\right) H_2^e \\ &= \left(\frac{\sum_{t=1}^{n} \beta^t - \mu \sum_{t=1}^{n} (\mu\beta)^t}{\sum_{t=1}^{n} \beta^t} \right) \overline{\theta} H_0 - \left(\overline{\theta} + \pi'(H_2^e)\right) H_0 \\ &= -\overline{\theta} H_0 \left(1 - \frac{\sum_{t=1}^{n} \beta^t - \mu \sum_{t=1}^{n} (\mu\beta)^t}{\sum_{t=1}^{n} \beta^t}\right) - \pi'(H_2^e) H_0 \\ &= -\overline{\theta} H_0 \left(\frac{\mu \sum_{t=1}^{n} (\mu\beta)^t}{\sum_{t=1}^{n} \beta^t}\right) - \pi'(H_2^e) H_0 < 0. \end{aligned}$$

Thus, on the interval $[H_2^e, H^o]$, $HS(H)$ is decreasing as required.

Now suppose that (A23) holds. There are two possibilities: $S(H_0) \leq -H_0 B'(H_0)$ and $S(H_0) > -H_0 B'(H_0)$. In the first possibility, the equilibrium and optimal debt levels coincide and so the claim holds. In the second possibility, to prove the claim requires showing that $H_0 S(H_0) > H^o S(H^o)$. Following the argument just made, it suffices to show that $dH_0 S(H_0)/dH \leq 0$. We have that

$$\frac{dH_0 S(H_0)}{dH} = (1-H_0)\overline{\theta} + B(H_0) + H_0 B'(H_0) - C(1-\beta) - \pi(H_0) - H_0\left(\overline{\theta} + \pi'(H_0)\right)$$

Given that (A23) holds, we know that

$$\begin{aligned}
& (1-H_0)\overline{\theta} + B(H_0) + H_0 B'(H_0) - C(1-\beta) - \pi(H_0) - H_0\left(\overline{\theta} + \pi'(H_0)\right) \\
\leq\; & C(1-\beta) + \pi(H_0) + \left(\frac{\sum_{t=1}^{n} \beta^t - \mu \sum_{t=1}^{n} (\mu\beta)^t}{\sum_{t=1}^{n} \beta^t}\right) \overline{\theta} H_0 - C(1-\beta) - \pi(H_0) - H_0\left(\overline{\theta} + \pi'(H_0)\right) \\
=\; & \left(\frac{\sum_{t=1}^{n} \beta^t - \mu \sum_{t=1}^{n} (\mu\beta)^t}{\sum_{t=1}^{n} \beta^t}\right) \overline{\theta} H_0 - H_0\left(\overline{\theta} + \pi'(H_0)\right) < 0.
\end{aligned}$$

■

Comparative statics on debt

Changing $\overline{\theta}$

As argued in the text, the sign of the change in debt is determined by the sign of

$$\frac{d\left(H^e S(H^e)\right)}{dH}\frac{\partial H^e}{\partial \overline{\theta}} + H^e \frac{\partial S(H^e)}{\partial \overline{\theta}}.$$

From (21), we have that

$$\frac{\partial S(H^e)}{\partial \overline{\theta}} = 1 - H^e > 0.$$

Moreover, from (23), H^e is implicitly defined by the equation

$$S(H^e) + H^e B'(H^e) - \left(\overline{\theta} + \pi'(H^e)\right)(H^e - H_0) - \left(\frac{\mu(1-\beta)+1}{1-\mu\beta}\right)(1-\mu)\overline{\theta} H_0 = 0.$$

Thus, we have that

$$\left[\tfrac{dS(H^e)}{dH} + B'(H^e) + H^e B''(H^e) - \left(\overline{\theta} + \pi'(H^e)\right) - \pi''(H^e)(H^e - H_0)\right] dH^e \\ + \left[1 - H^e - (H^e - H_0) - \left(\tfrac{\mu(1-\beta)+1}{1-\mu\beta}\right)(1-\mu)H_0\right] d\overline{\theta} = 0.$$

Furthermore, from (21), we have that

$$\begin{aligned} & \frac{dS(H^e)}{H} + B'(H^e) + H^e B''(H^e) - \left(\overline{\theta} + \pi'(H^e)\right) - \pi''(H^e)(H^e - H_0) \\ = \quad & -2\overline{\theta} + 2B'(H^e) - 2\pi'(H^e) + H^e B''(H^e). \end{aligned}$$

Thus,

$$\frac{\partial H^e}{\partial \overline{\theta}} = \frac{1 - H^e - (H^e - H_0) - \left(\frac{\mu(1-\beta)+1}{1-\mu\beta}\right)(1-\mu)H_0}{2\overline{\theta} - 2B'(H^e) + 2\pi'(H^e) - H^e B''(H^e)}$$

Finally, as shown in the proof of Proposition 4,

$$\frac{dH^e S(H^e)}{dH} = -\overline{\theta} H_0 \left(\frac{\mu^2(1-\beta)}{1-\mu\beta}\right) - \pi'(H^e)H_0 < 0.$$

Thus,

$$\begin{aligned} & \frac{d\left(H^e S(H^e)\right)}{dH}\frac{\partial H^e}{\partial \overline{\theta}} + H^e \frac{\partial S(H^e)}{\partial \overline{\theta}} \\ = \quad & \frac{-\left(1 - H^e - (H^e - H_0) - \left(\frac{\mu(1-\beta)+1}{1-\mu\beta}\right)(1-\mu)H_0\right) H_0 \left(\overline{\theta}\frac{\mu^2(1-\beta)}{1-\mu\beta} + \pi'(H^e)\right)}{2\overline{\theta} - 2B'(H^e) + 2\pi'(H^e) - H^e B''(H^e)} \\ & +1 - H^e \end{aligned}$$

$$
\begin{aligned}
&> \frac{-\left(1-H^e-(H^e-H_0)-\left(\frac{\mu(1-\beta)+1}{1-\mu\beta}\right)(1-\mu)H_0\right)H_0\left(\overline{\theta}+\pi'(H^e)\right)}{2\overline{\theta}+2\pi'(H^e)-H^eB''(H^e)-2B'(H^e)}+1-H^e \\
&> \frac{-\left(1-H^e-(H^e-H_0)-\left(\frac{\mu(1-\beta)+1}{1-\mu\beta}\right)(1-\mu)H_0\right)H_0\left(\overline{\theta}+\pi'(H^e)\right)}{2\overline{\theta}+2\pi'(H^e)}+1-H^e \\
&= \frac{-\left(1-H^e-(H^e-H_0)-\left(\frac{\mu(1-\beta)+1}{1-\mu\beta}\right)(1-\mu)H_0\right)H_0}{2}+1-H^e \\
&= \frac{1-H^e}{2}+\frac{\left((H^e-H_0)+\left(\frac{\mu(1-\beta)+1}{1-\mu\beta}\right)(1-\mu)H_0\right)H_0}{2}>0.
\end{aligned}
$$

Changing μ

The sign of the change in debt is determined by the sign of

$$
\frac{d\left(H^eS(H^e)\right)}{dH}\frac{\partial H^e}{\partial \mu}+H^e\frac{\partial S(H^e)}{\partial \mu}.
$$

From (21), we have that

$$
\frac{\partial S(H^e)}{\partial \mu}=0.
$$

Moreover, from (23), H^e is implicitly defined by the equation

$$
S(H^e)+H^eB'(H^e)-\left(\overline{\theta}+\pi'(H^e)\right)(H^e-H_0)-\left(\frac{\mu(1-\beta)+1}{1-\mu\beta}\right)(1-\mu)\overline{\theta}H_0=0.
$$

Thus, we have that

$$
\begin{aligned}
&\left[\frac{dS(H^e)}{dH}+B'(H^e)+H^eB''(H^e)-\left(\overline{\theta}+\pi'(H^e)\right)-\pi''(H^e)(H^e-H_0)\right]dH^e \\
&\qquad -\left[\frac{d\frac{(\mu(1-\beta)+1)(1-\mu)}{1-\mu\beta}}{d\mu}\right]\overline{\theta}H_0d\mu=0.
\end{aligned}
$$

We have that

$$
\frac{d\frac{(\mu(1-\beta)+1)(1-\mu)}{1-\mu\beta}}{d\mu}=-\frac{2\mu(1-\beta)}{1-\mu\beta},
$$

and, from (21), that

$$
\begin{aligned}
&\frac{dS(H^e)}{dH}+B'(H^e)+H^eB''(H^e)-\left(\overline{\theta}+\pi'(H^e)\right)-\pi''(H^e)(H^e-H_0) \\
=\; & -2\overline{\theta}+2B'(H^e)-2\pi'(H^e)+H^eB''(H^e).
\end{aligned}
$$

Thus,

$$
\frac{\partial H^e}{\partial \mu}=\frac{\frac{2\mu(1-\beta)}{1-\mu\beta}}{2\overline{\theta}-2B'(H^e)+2\pi'(H^e)-H^eB''(H^e)}>0.
$$

Finally, as shown in the proof of Proposition 4,

$$\frac{dH^e S(H^e)}{dH} = -\overline{\theta} H_0 (\frac{\mu^2(1-\beta)}{1-\mu\beta}) - \pi'(H^e) H_0 < 0.$$

Thus,

$$\begin{aligned} & \frac{d\left(H^e S(H^e)\right)}{dH} \frac{\partial H^e}{\partial \mu} + H^e \frac{\partial S(H^e)}{\partial \mu} \\ = \quad & \frac{-H_0 \left(\overline{\theta} \frac{\mu^2(1-\beta)}{1-\mu\beta} + \pi'(H^e) \right) \frac{2\mu(1-\beta)}{1-\mu\beta}}{2\overline{\theta} - 2B'(H^e) + 2\pi'(H^e) - H^e B''(H^e)} < 0. \end{aligned}$$

Changing C

The sign of the change in debt is determined by the sign of

$$\frac{d\left(H^e S(H^e)\right)}{dH} \frac{\partial H^e}{\partial C} + H^e \frac{\partial S(H^e)}{\partial C}.$$

From (21), we have that

$$\frac{\partial S(H^e)}{\partial C} = -(1-\beta).$$

Moreover, from (23), H^e is implicitly defined by the equation

$$S(H^e) + H^e B'(H^e) - \left(\overline{\theta} + \pi'(H^e)\right)(H^e - H_0) - \left(\frac{\mu(1-\beta)+1}{1-\mu\beta} \right)(1-\mu)\overline{\theta} H_0 = 0.$$

Thus,

$$\left[\frac{dS(H^e)}{H} + B'(H^e) + H^e B''(H^e) - \left(\overline{\theta} + \pi'(H^e)\right) - \pi''(H^e)(H^e - H_0) \right] dH^e - (1-\beta) dC = 0.$$

Furthermore, from (21), we have that

$$\begin{aligned} & \frac{dS(H^e)}{dH} + B'(H^e) + H^e B''(H^e) - \left(\overline{\theta} + \pi'(H^e)\right) - \pi''(H^e)(H^e - H_0) \\ = \quad & -2\overline{\theta} + 2B'(H^e) - 2\pi'(H^e) + H^e B''(H^e). \end{aligned}$$

Thus, we have that

$$\frac{\partial H^e}{\partial C} = -\frac{1-\beta}{2\overline{\theta} - 2B'(H^e) + 2\pi'(H^e) - H^e B''(H^e)} < 0.$$

Finally, as shown in the proof of Proposition 4,

$$\frac{dH^e S(H^e)}{dH} = -\overline{\theta} H_0 (\frac{\mu^2(1-\beta)}{1-\mu\beta}) - \pi'(H^e) H_0 < 0.$$

Thus,

$$
\begin{aligned}
& \frac{d\left(H^e S(H^e)\right)}{dH}\frac{\partial H^e}{\partial C} + H^e\frac{\partial S(H^e)}{\partial C} \\
= \quad & \frac{\left(\overline{\theta}H_0(\frac{\mu^2(1-\beta)}{1-\mu\beta}) + \pi'(H^e)H_0\right)(1-\beta)}{2\overline{\theta} - 2B'(H^e) + 2\pi'(H^e) - H^e B''(H^e)} - H^e(1-\beta) \\
= \quad & (1-\beta)\left[H_0\left(\frac{\overline{\theta}(\frac{\mu^2(1-\beta)}{1-\mu\beta}) + \pi'(H^e)}{2\overline{\theta} - 2B'(H^e) + 2\pi'(H^e) - H^e B''(H^e)}\right) - H^e\right] \\
< \quad & (1-\beta)\left[H_0\left(\frac{\overline{\theta} + \pi'(H^e)}{2\left(\overline{\theta} + 2\pi'(H^e)\right) - 2B'(H^e) - H^e B''(H^e)}\right) - H^e\right] \\
< \quad & (1-\beta)\left[H_0\left(\frac{\overline{\theta} + \pi'(H^e)}{2\left(\overline{\theta} + 2\pi'(H^e)\right)}\right) - H^e\right] \\
= \quad & (1-\beta)\left[\frac{H_0}{2} - H^e\right] < 0.
\end{aligned}
$$