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Appendix (For Online Publication)

to

Community Development by Public Wealth Accumulation

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Appendix 0: Properties of $\mathcal{H}(W)$ and $\mathcal{W}(H)$

Recall from Section 5, that $\mathcal{H}(W)$ was defined to be the largest housing level in the interval $[0,1]$ that satisfies the equality

$$(1-H)\,\overline{\theta}+S(H)+\frac{(1-\beta)W}{H}-C(1-\beta)=\underline{u}.$$

As noted, $\mathcal{H}(W)$ will not be well-defined if W is so large that all types of households would get a positive benefit from living in the community or if W is so small (i.e., sufficiently negative) that there is no population size at which residents would be willing to pay a price of housing C to live in the community.

To obtain the upper bound on W, note first that if

$$S(1)+(1-\beta)W-C(1-\beta)>\underline{u},$$

all types would get a positive benefit from living in the community. Thus, the upper bound on W is

$$\overline{W}\equiv\frac{\underline{u}+C(1-\beta)-S(1)}{1-\beta}.$$

To obtain the lower bound on W, consider the problem

$$\max_{H}\left\{(1-H)\,\overline{\theta}+S(H)+\frac{(1-\beta)W}{H}-C(1-\beta)\right\}.$$

Let the solution be denoted $H(W)$ and let $\underline{W}$ be the wealth level such that

$$(1-H(\underline{W}))\,\overline{\theta}+S(H(\underline{W}))+\frac{(1-\beta)\underline{W}}{H(\underline{W})}-C(1-\beta)=\underline{u}.$$

The lower bound is $\underline{W}$. To characterize it in a more useful way, note that the first order condition for $H(W)$ is

$$-\overline{\theta}+S'(H)-\frac{(1-\beta)W}{H^{2}}=0,$$

which implies that

$$\frac{(1-\beta)W}{H(W)}=-\overline{\theta}H(W)+H(W)S'(H(W)).$$

It follows that

$$(1-H(\underline{W}))\,\overline{\theta}+S(H(\underline{W}))+\frac{(1-\beta)\underline{W}}{H(\underline{W})}-C(1-\beta)=(1-2H(\underline{W}))\,\overline{\theta}+S(H(\underline{W}))+H(\underline{W})S'(H(\underline{W}))-C(1-\beta).$$

Thus, if we define the housing level $\underline{H}$ as the solution to:

$$(1-\underline{H})\,\overline{\theta}+S(\underline{H})+\underline{H}S'(\underline{H})-C(1-\beta)=\underline{u}+\underline{H}\overline{\theta},$$

then the lower bound can be written as

$$\underline{W} = -\frac{\underline{H}\left[(1-\underline{H})\,\overline{\theta} + S(\underline{H}) - C(1-\beta) - \underline{u}\right]}{1-\beta}.$$

On the interval $[\underline{W}, \overline{W}]$, it is straightforward to verify that

$$\mathcal{H}'(W) = -\frac{1-\beta}{(1-2\mathcal{H}(W))\,\overline{\theta} + S(\mathcal{H}(W)) + \mathcal{H}(W)S'(\mathcal{H}(W)) - C(1-\beta) - \underline{u}}. \tag{A1}$$

This is positive because the denominator must be positive given Assumption 1(i) and the fact that $\mathcal{H}(W) > \underline{H}$. Moreover,

$$\mathcal{H}''(W) = \frac{(1-\beta)\left[-2\overline{\theta} + 2S'(\mathcal{H}(W)) + \mathcal{H}(W)S''(\mathcal{H}(W))\right]\mathcal{H}'(W)}{\left[(1-2\mathcal{H}(W))\,\overline{\theta} + S(\mathcal{H}(W)) + \mathcal{H}(W)S'(\mathcal{H}(W)) - C(1-\beta) - \underline{u}\right]^2},$$

which is negative by Assumption 1(i). Thus, $\mathcal{H}(W)$ is increasing and concave on the interval $[\underline{W}, \overline{W}]$.

Because $\mathcal{H}(W)$ is increasing on the interval $[\underline{W}, \overline{W}]$, it has an inverse, $\mathcal{W}(H)$, defined on the interval $[\mathcal{H}(\underline{W}), \mathcal{H}(\overline{W})] = [\underline{H}, 1]$. Note that the first inequality of Assumption 2 implies that $\underline{H} < H_0$ which means that $\mathcal{W}(H)$ is well-defined on the entire interval $[H_0, 1]$. In light of the properties of $\mathcal{H}(W)$, $\mathcal{W}(H)$ will be increasing and convex. Note for future reference that

$$\mathcal{W}'(H) = -\frac{(1-2H)\,\overline{\theta} + S(H) + HS'(H) - C(1-\beta) - \underline{u}}{1-\beta}. \tag{A2}$$

Appendix 1: Proof of Propositions 2 and 3

Both these results concern the nature of the solution to the problem faced by the period t residents. It is helpful to begin with a clear statement of that problem.

Statement of the problem

The period t residents' problem is

$$\max_{\{g_{\tau+1},b_{\tau+1},T_\tau,H_{\tau+1},P_\tau\}_{\tau=t}^{\infty}} \left\{ \begin{array}{c} \sum_{\tau=t}^{\infty} (\mu\beta)^{\tau-t} \left\{(1-\mu)\left[P_\tau + \frac{\underline{u}}{1-\beta}\right] + \mu\left[B\left(\frac{g_{\tau+1}/(1-\delta)}{(H_{\tau+1})^{\alpha}}\right) - T_\tau\right]\right\} \\ s.t. \text{ for all periods } \tau \\ (1+\rho)b_\tau + c\left(\frac{g_{\tau+1}}{1-\delta} - g_\tau\right) = b_{\tau+1} + H_{\tau+1}T_\tau \\ P_\tau = (1-H_{\tau+1})\overline{\theta} + B\left(\frac{g_{\tau+1}/(1-\delta)}{(H_{\tau+1})^{\alpha}}\right) - T_\tau + \beta P_{\tau+1} - \underline{u} \\ H_{\tau+1} \geq H_\tau \\ P_\tau \leq C \;(\,=\text{if } H_{\tau+1} > H_\tau) \end{array} \right\}. \tag{A3}$$

The first constraint is the budget constraint. The remaining three constraints are the market equilibrium constraints. The residents also need to respect the transversality condition that $\lim_{\tau\to\infty} \beta^\tau b_\tau = 0$, to prevent them operating a Ponzi scheme. Finally, the residents face initial conditions (g_t, b_t, H_t). We assume that $H_t \in [H_0, 1]$ and that $W_t \leq \mathcal{W}(1)$.

Solving the problem

Problem (A3) involves too many constraints to tackle head on by forming a Lagrangian. Rather, we must approach it through a process of simplification. Our first result concerns the objective function.

Fact A.1.1. *Suppose that the sequence of policies $\{g_{\tau+1}, b_{\tau+1}, T_\tau, H_{\tau+1}, P_\tau\}_{\tau=t}^{\infty}$ satisfies in each period τ the market equilibrium conditions. Then, the period t residents' objective function is equal to*

$$P_t + \mu\overline{\theta}\sum_{\tau=t}^{\infty} (\mu\beta)^{\tau-t}(H_{\tau+1} - 1) + \frac{\underline{u}}{1-\beta}. \tag{A4}$$

Proof of Fact A.1.1. From the market equilibrium condition

$$P_\tau = (1-H_{\tau+1})\overline{\theta} + B\left(\frac{g_{\tau+1}/(1-\delta)}{(H_{\tau+1})^{\alpha}}\right) - T_\tau + \beta P_{\tau+1} - \underline{u}, \tag{A5}$$

we have that

$$B\left(\frac{g_{\tau+1}/(1-\delta)}{(H_{\tau+1})^{\alpha}}\right)-T_{\tau}=P_{\tau}+(H_{\tau+1}-1)\overline{\theta}-\beta P_{\tau+1}+\underline{u}.$$

Thus, for any period $\tau \geq t$, we have that

$$\begin{aligned}
&\sum_{z=t}^{\tau}(\mu\beta)^{z-t}\left\{(1-\mu)\left[P_z+\frac{\underline{u}}{1-\beta}\right]+\mu\left[B\left(\frac{g_{z+1}/(1-\delta)}{(H_{z+1})^{\alpha}}\right)-T_z\right]\right\}\\
=\;&\sum_{z=t}^{\tau}(\mu\beta)^{z-t}\left\{(1-\mu)\left[P_z+\frac{\underline{u}}{1-\beta}\right]+\mu\left[P_z+(H_{z+1}-1)\overline{\theta}-\beta P_{z+1}+\underline{u}\right]\right\}.
\end{aligned}$$

Expanding the right hand side, we have that

$$\begin{aligned}
&\sum_{z=t}^{\tau}(\mu\beta)^{z-t}\left\{(1-\mu)\left[P_z+\frac{\underline{u}}{1-\beta}\right]+\mu\left[P_z+(H_{z+1}-1)\overline{\theta}-\beta P_{z+1}+\underline{u}\right]\right\}\\
=\;&(1-\mu)\left[P_t+\frac{\underline{u}}{1-\beta}\right]+\mu\left[P_t+(H_{t+1}-1)\overline{\theta}-\beta P_{t+1}+\underline{u}\right]\\
&+\mu\beta\left\{(1-\mu)\left[P_{t+1}+\frac{\underline{u}}{1-\beta}\right]+\mu\left[P_{t+1}+(H_{t+2}-1)\overline{\theta}-\beta P_{t+2}+\underline{u}\right]\right\}\\
&+(\mu\beta)^2\left\{(1-\mu)\left[P_{t+2}+\frac{\underline{u}}{1-\beta}\right]+\mu\left[P_{t+2}+(H_{t+3}-1)\overline{\theta}-\beta P_{t+3}+\underline{u}\right]\right\}\\
&+\ldots..+(\mu\beta)^{\tau-t}\left\{(1-\mu)\left[P_{\tau}+\frac{\underline{u}}{1-\beta}\right]+\mu\left[P_{\tau}+(H_{\tau+1}-1)\overline{\theta}-\beta P_{\tau+1}+\underline{u}\right]\right\}.
\end{aligned}$$

Note that the P_{t+1} term in the first line cancels with that in the second line. Similarly, the P_{t+2} term in the second line cancels with the P_{t+2} term in the third line, etc, etc. Thus, we have

$$\begin{aligned}
&\sum_{z=t}^{\tau}(\mu\beta)^{z-t}\left\{(1-\mu)\left[P_z+\frac{\underline{u}}{1-\beta}\right]+\mu\left[P_z+(H_{z+1}-1)\overline{\theta}-\beta P_{z+1}+\underline{u}\right]\right\}\\
=\;&P_t+\sum_{z=t}^{\tau}(\mu\beta)^{z-t}\left\{(1-\mu)\frac{\underline{u}}{1-\beta}+\mu\left[(H_{z+1}-1)\overline{\theta}+\underline{u}\right]\right\}-(\mu\beta)^{\tau+1-t}P_{\tau+1}.
\end{aligned}$$

Given that the third market equilibrium condition implies that $P_{\tau} \leq C$ for all τ, we have that $\lim_{\tau\to\infty}(\mu\beta)^{\tau+1-t}P_{\tau+1}=0$. Thus,

$$\begin{aligned}
&\sum_{\tau=t}^{\infty}(\mu\beta)^{\tau-t}\left\{(1-\mu)\left[P_{\tau}+\frac{\underline{u}}{1-\beta}\right]+\mu\left[P_{\tau}+(H_{\tau+1}-1)\overline{\theta}-\beta P_{\tau+1}+\underline{u}\right]\right\}\\
=\;&P_t+\mu\overline{\theta}\sum_{\tau=t}^{\infty}(\mu\beta)^{\tau-t}(H_{\tau+1}-1)+\frac{\underline{u}}{1-\beta}.
\end{aligned}$$

■

This result tells us that housing prices have a direct impact on the objective function only in the initial period. Prices in all subsequent periods wash out. Moreover, all else equal, the initial residents prefer to have future housing levels as high as possible. This reflects that, when the size of the community is determined by the market, higher housing levels correspond to greater surplus from living in the community.

Our second result concerns the inter-temporal implications of the sequence of budget constraints.

Fact A.1.2. *Suppose that the sequence of policies* $\{g_{\tau+1}, b_{\tau+1}, T_\tau, H_{\tau+1}\}_{\tau=t}^{\infty}$ *satisfies in each period* τ *the budget constraint*

$$(1+\rho)b_\tau + c\left(\frac{g_{\tau+1}}{1-\delta} - g_\tau\right) = b_{\tau+1} + H_{\tau+1}T_\tau \quad \text{(A6)}$$

and that $\lim_{\tau\to\infty}\beta^\tau g_\tau = \lim_{\tau\to\infty}\beta^\tau b_\tau = 0$. *Then,* $\{g_{\tau+1}, T_\tau, H_{\tau+1}\}_{\tau=t}^{\infty}$ *satisfies the intertemporal budget constraint*

$$\sum_{\tau=t}^{\infty}\beta^{\tau-t}\left[c\left(\frac{g_{\tau+1}}{1-\delta} - \beta g_{\tau+1}\right) - H_{\tau+1}T_\tau\right] = W_t. \quad \text{(A7)}$$

Proof of Fact A.1.2. Suppose that the sequence of policies $\{g_{\tau+1}, b_{\tau+1}, T_\tau, H_{\tau+1}\}_{\tau=t}^{\infty}$ satisfies in each period τ the budget constraint (A6). From the period t budget constraint, we have that

$$c\frac{g_{t+1}}{1-\delta} - b_{t+1} - H_{t+1}T_t = W_t,$$

and from the period $t+1$ budget constraint, we have that

$$b_{t+1} = \beta\left(b_{t+2} + H_{t+2}T_{t+1} - c\left(\frac{g_{t+2}}{1-\delta} - g_{t+1}\right)\right).$$

Substituting the latter into the former yields

$$c\left(\frac{g_{t+1}}{1-\delta} - \beta g_{t+1}\right) + \beta c\frac{g_{t+2}}{1-\delta} - \beta b_{t+2} - H_{t+1}T_t - \beta H_{t+2}T_{t+1} = W_t.$$

Similarly, the period $t+2$ budget constraint tells us that

$$b_{t+2} = \beta\left(b_{t+3} + H_{t+3}T_{t+2} - c\left(\frac{g_{t+3}}{1-\delta} - g_{t+2}\right)\right).$$

Substituting this into the period t budget constraint yields

$$c\left(\frac{g_{t+1}}{1-\delta} - \beta g_{t+1}\right) + \beta\left(c\frac{g_{t+2}}{1-\delta} - \beta g_{t+2}\right) + \beta^2 c\frac{g_{t+3}}{1-\delta} - \beta^2 b_{t+3} - H_{t+1}T_t - \beta H_{t+2}T_{t+1} - \beta^2 H_{t+3}T_{t+2} = W_t.$$

Iterating this logic, we find that for all periods $\tau \geq t$

$$\sum_{z=t}^{\tau}\beta^{z-t}\left[c\left(\frac{g_{z+1}}{1-\delta} - \beta g_{z+1}\right) - H_{z+1}T_z\right] + \beta^{\tau-t}\left(cg_{\tau+1} - b_{\tau+1}\right) = W_t$$

Since $\lim_{\tau\to\infty}\beta^{\tau-t}\left(cg_{\tau+1} - b_{\tau+1}\right) = 0$, this implies that

$$\sum_{\tau=t}^{\infty}\beta^{\tau-t}\left[c\left(\frac{g_{\tau+1}}{1-\delta} - \beta g_{\tau+1}\right) - H_{\tau+1}T_\tau\right] = W_t,$$

which is (A7). ■

The assumed properties of the public good benefit function B imply that the public good level will be bounded above and hence that $\lim_{\tau\to\infty}\beta^{\tau}g_{\tau}=0$. Moreover, the transversality condition requires that $\lim_{\tau\to\infty}\beta^{\tau}b_{\tau}=0$. Thus, Fact A.1.2 suggests replacing the sequence of budget constraints (A6) in the period t residents' problem with the single intertemporal budget constraint (A7). Indeed, this is a standard procedure in models of optimal policy in which decision-makers face a sequence of budget constraints and have access to bonds. However, in our problem, we also have the market equilibrium constraints to worry about and these depend on the time path of taxes and hence local government debt. Specifically, while constraint (A7) is independent of the time path of debt and just depends on the present value of taxes, the market equilibrium conditions (A5) do depend on the time path of taxes. Thus, we need to verify that replacing the sequence of budget constraints with the single intertemporal budget constraint would not create problems in satisfying the market equilibrium conditions (A5). This is confirmed by our next result.

Fact A.1.3. *Suppose that the sequence of policies* $\{g_{\tau+1},b_{\tau+1},T_{\tau},H_{\tau+1},P_{\tau}\}_{\tau=t}^{\infty}$ *satisfies in each period* τ *the market equilibrium condition (A5) and the budget constraint (A6) and that* $\lim_{\tau\to\infty}\beta^{\tau}g_{\tau}=\lim_{\tau\to\infty}\beta^{\tau}b_{\tau}=0$. *Then,* $\{g_{\tau+1},b_{\tau+1},H_{\tau+1},P_{\tau}\}_{\tau=t}^{\infty}$ *satisfies the constraints that*

$$\sum_{\tau=t}^{\infty}\beta^{\tau-t}\left[c\left(\frac{g_{\tau+1}}{1-\delta}-\beta g_{\tau+1}\right)-H_{\tau+1}\left((1-H_{\tau+1})\overline{\theta}+B\left(\frac{g_{\tau+1}/(1-\delta)}{(H_{\tau+1})^{\alpha}}\right)-P_{\tau}+\beta P_{\tau+1}-\underline{u}\right)\right]=W_t, \tag{A8}$$

and that, for all periods τ,

$$P_{\tau}=(1-H_{\tau+1})\overline{\theta}+B\left(\frac{g_{\tau+1}/(1-\delta)}{(H_{\tau+1})^{\alpha}}\right)-\left(\frac{(1+\rho)b_{\tau}+c\left(\frac{g_{\tau+1}}{1-\delta}-g_{\tau}\right)-b_{\tau+1}}{H_{\tau+1}}\right)+\beta P_{\tau+1}-\underline{u}. \tag{A9}$$

Conversely, if $\{g_{\tau+1},b_{\tau+1},H_{\tau+1},P_{\tau}\}_{\tau=t}^{\infty}$ *satisfies the constraints (A8) and, for all* τ, *(A9), then there exists* $\{T_{\tau}\}_{\tau=t}^{\infty}$ *such that* $\{g_{\tau+1},b_{\tau+1},T_{\tau},H_{\tau+1},P_{\tau}\}_{\tau=t}^{\infty}$ *satisfies for all* τ *the market equilibrium condition (A5) and the budget constraint (A6).*

Proof of Fact A.1.3. Let $\{g_{\tau+1},b_{\tau+1},T_{\tau},H_{\tau+1},P_{\tau}\}_{\tau=t}^{\infty}$ be a sequence of policies satisfying in each period τ the market equilibrium condition (A5), the budget constraint (A6) and the requirement that $\lim_{\tau\to\infty}\beta^{\tau}g_{\tau}=\lim_{\tau\to\infty}\beta^{\tau}b_{\tau}=0$. Then we know from Fact A.1.2 that

$$\sum_{\tau=t}^{\infty}\beta^{\tau-t}\left[c\left(\frac{g_{\tau+1}}{1-\delta}-\beta g_{\tau+1}\right)-H_{\tau+1}T_{\tau}\right]=W_t. \tag{A10}$$

Using (A5) to solve for T_τ and substituting this into (A10) yields (A8). That (A9) holds follows immediately from (A5) after solving (A6) for T_τ and substituting in for T_τ.

For the converse, let $\{g_{\tau+1}, b_{\tau+1}, H_{\tau+1}, P_\tau\}_{\tau=t}^{\infty}$ satisfy the constraints (A8) and, for all $\tau \geq t$, (A9). For all τ, let

$$T_\tau = (1 - H_{\tau+1})\overline{\theta} + B\left(\frac{g_{\tau+1}/(1-\delta)}{(H_{\tau+1})^{\alpha}}\right) - P_\tau + \beta P_{\tau+1} - \underline{u}. \tag{A11}$$

Simply rearranging this equation reveals that $\{g_{\tau+1}, b_{\tau+1}, T_\tau, H_{\tau+1}, P_\tau\}_{\tau=t}^{\infty}$ satisfies (A5) for all τ. Solving (A11) for P_τ and using (A9) reveals that, for all periods τ,

$$0 = T_\tau - \left(\frac{(1+\rho)b_\tau + c\left(\frac{g_{\tau+1}}{1-\delta} - g_\tau\right) - b_{\tau+1}}{H_{\tau+1}}\right).$$

Rearranging this equation yields (A6). ■

Note that equation (A8) is obtained from the intertemporal budget constraint (A7) by substituting in the expression for the tax rate implied by the market equilibrium condition. Equation (A9) is obtained from the market equilibrium condition by substituting in the expression for the tax rate that is implied by the budget constraint. Thus, both equations reflect both the market equilibrium conditions and the budget constraint. Fact A.1.3 tells us that we can replace the per-period budget constraint and the market equilibrium condition with equations (A8) and (A9). It also allows us to eliminate T_t from the set of choice variables. Thus, we can recast the initial residents' problem as follows:

$$\max_{\{g_{\tau+1}, b_{\tau+1}, H_{\tau+1}, P_\tau\}_{\tau=t}^{\infty}} \left\{ \begin{array}{c} P_t + \mu\overline{\theta}\sum_{\tau=t}^{\infty}(\mu\beta)^{\tau-t}(H_{\tau+1} - 1) + \frac{\underline{u}}{1-\beta} \\ s.t.\ (A8) \text{ and for all } \tau \ (A9),\ H_{\tau+1} \geq H_t,\ P_\tau \leq C\ (= \text{if } H_{\tau+1} > H_\tau) \end{array} \right\} \tag{A12}$$

Our next result shows that there is no loss of generality in requiring the period t residents choose policies so that the price of housing is equal to C in all periods except period t.

Fact A.1.4. *Let $\{g_{\tau+1}, b_{\tau+1}, H_{\tau+1}, P_\tau\}_{\tau=t}^{\infty}$ be a sequence of policies satisfying the constraints (A8) and, for all τ, (A9). Then, there exists $\{\widetilde{b}_\tau, \widetilde{P}_\tau\}_{\tau=t}^{\infty}$ such that $\widetilde{P}_\tau = C$ for all $\tau \geq t+1$, $\widetilde{P}_t = P_t$, and $\widetilde{b}_t = b_t$, with the property that $\{g_{\tau+1}, \widetilde{b}_{\tau+1}, H_{\tau+1}, \widetilde{P}_\tau\}_{\tau=t}^{\infty}$ satisfies the constraints (A8) and, for all τ, (A9).*

Proof of Fact A.1.4. Let $\{g_{\tau+1}, b_{\tau+1}, H_{\tau+1}, P_\tau\}_{\tau=t}^{\infty}$ be a sequence of policies satisfying the constraints (A8) and, for all τ, (A9). Let $\{\widetilde{P}_\tau\}_{\tau=t}^{\infty}$ be such that $\widetilde{P}_\tau = C$ for all $\tau \geq t+1$ and

$\widetilde{P}_t = P_t$. Then we claim that $\{g_{\tau+1}, H_{\tau+1}, \widetilde{P}_\tau\}_{\tau=t}^{\infty}$ satisfies the intertemporal budget constraint (A8). To prove this, it suffices to show that

$$\sum_{\tau=t}^{\infty}\beta^{\tau-t}\left[c\left(\tfrac{g_{\tau+1}}{1-\delta}-\beta g_{\tau+1}\right)-H_{\tau+1}\left((1-H_{\tau+1})\overline{\theta}+B\left(\tfrac{g_{\tau+1}/(1-\delta)}{(H_{\tau+1})^{\alpha}}\right)-P_\tau+\beta P_{\tau+1}-\underline{u}\right)\right]=$$

$$\left[c\left(\tfrac{g_{t+1}}{1-\delta}-\beta g_{t+1}\right)-H_{t+1}\left((1-H_{t+1})\overline{\theta}+B\left(\tfrac{g_{t+1}/(1-\delta)}{(H_{t+1})^{\alpha}}\right)-P_t+\beta C-\underline{u}\right)\right]+$$
$$\sum_{\tau=t+1}^{\infty}\beta^{\tau-t}\left[c\left(\tfrac{g_{\tau+1}}{1-\delta}-\beta g_{\tau+1}\right)-H_{\tau+1}\left((1-H_{\tau+1})\overline{\theta}+B\left(\tfrac{g_{\tau+1}/(1-\delta)}{(H_{\tau+1})^{\alpha}}\right)-C(1-\beta)-\underline{u}\right)\right].$$

For this, we need to show that

$$\sum_{\tau=t}^{\infty}\beta^{\tau-t}H_{\tau+1}\left(P_\tau-\beta P_{\tau+1}\right)=H_{t+1}\left(P_t-\beta C\right)+\sum_{\tau=t+1}^{\infty}\beta^{\tau-t}H_{\tau+1}C\left(1-\beta\right).$$

We have

$$\sum_{\tau=t}^{\infty}\beta^{\tau-t}H_{\tau+1}\left(P_\tau-\beta P_{\tau+1}\right)=H_{t+1}\left(P_t-\beta P_{t+1}\right)+\beta H_{t+2}\left(P_{t+1}-\beta P_{t+2}\right)+\beta^2 H_{t+3}\left(P_{t+2}-\beta P_{t+3}\right)+...$$

Let $z\geq t+1$ be the first time period in which $P_z<C$. Then, $H_z=H_{z+1}$. Thus,

$$\begin{aligned}
&\beta^{z-1-t}H_z\left(C-\beta P_z\right)+\beta^{z-t}H_{z+1}\left(P_z-\beta P_{z+1}\right)\\
=\ &\beta^{z-1-t}H_z\left[C-\beta P_z+\beta\left(P_z-\beta P_{z+1}\right)\right]\\
=\ &\beta^{z-1-t}H_z\left(C-\beta C\right)+\beta^{z-t}H_{z+1}\left(C-\beta P_{z+1}\right).
\end{aligned}$$

Continuing this argument yields the result.

Now define the sequence of bond levels $\{\widetilde{b}_{\tau+1}\}_{\tau=t}^{\infty}$ inductively as follows:

$$\widetilde{b}_{t+1}=c\frac{g_{t+1}}{1-\delta}-W_t-H_{t+1}\left((1-H_{t+1})\overline{\theta}+B\left(\frac{g_{t+1}/(1-\delta)}{(H_{t+1})^{\alpha}}\right)-P_t+\beta C-\underline{u}\right)$$

and

$$\widetilde{b}_\tau=(1+\rho)\widetilde{b}_{\tau-1}+c\left(\frac{g_\tau}{1-\delta}-g_{\tau-1}\right)-H_\tau\left((1-H_\tau)\overline{\theta}+B\left(\frac{g_\tau/(1-\delta)}{(H_\tau)^{\alpha}}\right)-C(1-\beta)-\underline{u}\right).$$

Then we have that

$$P_t=(1-H_{t+1})\overline{\theta}+B\left(\frac{g_{t+1}/(1-\delta)}{(H_{t+1})^{\alpha}}\right)-\left(\frac{(1+\rho)b_t+c\left(\frac{g_{t+1}}{1-\delta}-g_t\right)-\widetilde{b}_{t+1}}{H_{t+1}}\right)+\beta C-\underline{u}$$

and, for all $\tau\geq t+1$

$$C=(1-H_{\tau+1})\overline{\theta}+B\left(\frac{g_{\tau+1}/(1-\delta)}{(H_{\tau+1})^{\alpha}}\right)-\left(\frac{(1+\rho)\widetilde{b}_\tau+c\left(\frac{g_{\tau+1}}{1-\delta}-g_\tau\right)-\widetilde{b}_{\tau+1}}{H_{\tau+1}}\right)+\beta C-\underline{u}.$$

■

To understand this key result heuristically, imagine that, say, P_{t+1} were less than C. Then it must be the case that there is no new construction in period $t+1$, so that H_{t+2} equals H_{t+1}. The market equilibrium condition (A5) tells us that P_{t+1} will depend on the period $t+1$ tax T_{t+1}. Moreover, the government's budget constraint (A6) tells us that this tax will depend positively on the community's period $t+1$ debt b_{t+1}. Now suppose that in period t tax T_t were increased to reduce b_{t+1} sufficiently to raise P_{t+1} to C, with population held constant at H_{t+1}. The key point to note is that this will not change the housing market equilibrium in period t. In the market equilibrium condition (A5), the increase in taxes this period will be perfectly compensated by an increase in the value of housing next period. In this way, the time path of debt and taxes can be adjusted to make P_{t+1} equal to C without impacting P_t or the time path of housing. Given Fact A.1.1, this adjustment has no impact on the period t residents' payoff.

Fact A.1.4 allows us to write the intertemporal budget constraint (A8) as

$$c\left(\tfrac{g_{t+1}}{1-\delta}-\beta g_{t+1}\right)-H_{t+1}\left((1-H_{t+1})\overline{\theta}+B\left(\tfrac{g_{t+1}/(1-\delta)}{(H_{t+1})^{\alpha}}\right)-P_t+\beta C-\underline{u}\right)+$$
$$\sum_{\tau=t+1}^{\infty}\beta^{\tau-t}\left[c\left(\tfrac{g_{\tau+1}}{1-\delta}-\beta g_{\tau+1}\right)-H_{\tau+1}\left((1-H_{\tau+1})\overline{\theta}+B\left(\tfrac{g_{\tau+1}/(1-\delta)}{(H_{\tau+1})^{\alpha}}\right)-C(1-\beta)-\underline{u}\right)\right]=W_t. \tag{A13}$$

The market equilibrium constraints (A9) can be written as

$$P_t=(1-H_{t+1})\overline{\theta}+B\left(\frac{g_{t+1}/(1-\delta)}{(H_{t+1})^{\alpha}}\right)-\left(\frac{(1+\rho)b_t+c\left(\frac{g_{t+1}}{1-\delta}-g_t\right)-b_{t+1}}{H_{t+1}}\right)+\beta C-\underline{u}, \tag{A14}$$

and for all $\tau\geq t+1$

$$C=(1-H_{\tau+1})\overline{\theta}+B\left(\frac{g_{\tau+1}/(1-\delta)}{(H_{\tau+1})^{\alpha}}\right)-\left(\frac{(1+\rho)b_\tau+c\left(\frac{g_{\tau+1}}{1-\delta}-g_\tau\right)-b_{\tau+1}}{H_{\tau+1}}\right)+\beta C-\underline{u}. \tag{A15}$$

Finally, the constraint that $P_\tau\leq C$ (with equality if $H_{\tau+1}>H_\tau$) need only be imposed for period t.

Next observe that given a choice of period $t+1$ debt, b_{t+1}, and a sequence of public good and housing levels $\{g_{\tau+1},H_{\tau+1}\}_{\tau=t}^{\infty}$, the market equilibrium constraints (A15) pin down the sequence of debt levels $\{b_{\tau+1}\}_{\tau=t+1}^{\infty}$. Thus, these constraints can be eliminated and the debt levels $\{b_{\tau+1}\}_{\tau=t+1}^{\infty}$ can be removed from the set of choice variables. This allows us to write the period t residents' problem as:

$$\max_{\{P_t,b_{t+1},\{g_{\tau+1},H_{\tau+1}\}_{\tau=t}^{\infty}\}}\left\{\begin{array}{c} P_t+\mu\overline{\theta}\sum_{\tau=t}^{\infty}(\mu\beta)^{\tau-t}(H_{\tau+1}-1)+\frac{\underline{u}}{1-\beta} \\ s.t.\ c\left(\frac{g_{t+1}}{1-\delta}-\beta g_{t+1}\right)- \\ H_{t+1}\left((1-H_{t+1})\overline{\theta}+B\left(\frac{g_{t+1}/(1-\delta)}{(H_{t+1})^{\alpha}}\right)-P_t+\beta C-\underline{u}\right)+ \\ \sum_{\tau=t+1}^{\infty}\beta^{\tau-t}\left[\begin{array}{c} c\left(\frac{g_{\tau+1}}{1-\delta}-\beta g_{\tau+1}\right) \\ -H_{\tau+1}\left((1-H_{\tau+1})\overline{\theta}+B\left(\frac{g_{\tau+1}/(1-\delta)}{(H_{\tau+1})^{\alpha}}\right)-C(1-\beta)-\underline{u}\right)\end{array}\right]=W_t \\ P_t=(1-H_{t+1})\overline{\theta}+B\left(\frac{g_{t+1}/(1-\delta)}{(H_{t+1})^{\alpha}}\right)-\left(\frac{(1+\rho)b_t+c\left(\frac{g_{t+1}}{1-\delta}-g_t\right)-b_{t+1}}{H_{t+1}}\right)+\beta C-\underline{u} \\ H_{\tau+1}\geq H_\tau \text{ for all } \tau \\ P_t\leq C\ (=\text{if } H_{t+1}>H_t)\end{array}\right\} \tag{A16}$$

Our next result uses this formulation to tie down the public good levels.

Fact A.1.5. *In the period t residents' optimal plan, for all $\tau \geq t$*

$$g_{\tau+1}=(1-\delta)g^o(H_{\tau+1}).$$

Proof of Fact A.1.5. Inspecting problem (A16), it is clear that for all periods $\tau \geq t+1$, $g_{\tau+1}$ must equal $(1-\delta)g^o(H_{\tau+1})$. The only place $g_{\tau+1}$ enters the problem is in the intertemporal budget constraint and setting $g_{\tau+1}$ equal to the level $(1-\delta)g^o(H_{\tau+1})$ maximally relaxes this constraint. Given this, we can eliminate $\{g_{\tau+1}\}_{\tau=t+1}^{\infty}$ from the choice variables and use the definition of $\mathcal{W}(H_{\tau+1})$ to write the intertemporal budget constraint as:

$$c\left(\frac{g_{t+1}}{1-\delta}-\beta g_{t+1}\right)-H_{t+1}\left((1-H_{t+1})\overline{\theta}+B\left(\frac{g_{t+1}/(1-\delta)}{(H_{t+1})^{\alpha}}\right)-P_t+\beta C-\underline{u}\right) + \sum_{\tau=t+1}^{\infty}\beta^{\tau-t}\mathcal{W}(H_{\tau+1})(1-\beta)=W_t.$$

In addition, recalling that $W_\tau = cg_\tau-(1+\rho)b_\tau$, we can write the period t equilibrium market equilibrium constraint as

$$P_t=(1-H_{t+1})\overline{\theta}+B\left(\frac{g_{t+1}/(1-\delta)}{(H_{t+1})^{\alpha}}\right)-\left(\frac{c\frac{g_{t+1}}{1-\delta}-\beta cg_{t+1}}{H_{t+1}}\right)+\left(\frac{W_t-\beta W_{t+1}}{H_{t+1}}\right)+\beta C-\underline{u}.$$

Substituting this into the intertemporal budget constraint, we get

$$\sum_{\tau=t+1}^{\infty}\beta^{\tau-t}\mathcal{W}(H_{\tau+1})(1-\beta)=\beta W_{t+1}.$$

Thus, we can rewrite problem (A16) as

$$\max_{\{P_t, W_{t+1}, g_{t+1}, \{H_{\tau+1}\}_{\tau=t}^{\infty}\}} \left\{ \begin{array}{c} P_t + \mu\overline{\theta} \sum_{\tau=t}^{\infty} (\mu\beta)^{\tau-t} (H_{\tau+1} - 1) + \frac{\underline{u}}{1-\beta} \\ s.t. \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} \mathcal{W}(H_{\tau+1})(1-\beta) = \beta W_{t+1} \\ P_t = (1 - H_{t+1})\overline{\theta} + B\left(\frac{g_{t+1}/(1-\delta)}{(H_{t+1})^{\alpha}}\right) \\ -\left(\frac{c\frac{g_{t+1}}{1-\delta} - \beta c g_{t+1}}{H_{t+1}}\right) + \left(\frac{W_t - \beta W_{t+1}}{H_{t+1}}\right) + \beta C - \underline{u} \\ H_{\tau+1} \geq H_\tau \text{ for all } \tau \\ P_t \leq C \ (= \text{if } H_{t+1} > H_t) \end{array} \right\} \quad \text{(A17)}$$

We can now prove that g_{t+1} must equal $(1-\delta)g^o(H_{t+1})$. It is clear that this must be true if $P_t < C$, since then the optimal g_{t+1} must maximize P_t and hence

$$B\left(\frac{g_{t+1}/(1-\delta)}{(H_{t+1})^{\alpha}}\right) - \left(\frac{c\frac{g_{t+1}}{1-\delta} - \beta c g_{t+1}}{H_{t+1}}\right).$$

In this case, we have that

$$P_t = (1 - H_{t+1})\overline{\theta} + S(H_{t+1}) + \left(\frac{W_t - \beta W_{t+1}}{H_{t+1}}\right) + \beta C - \underline{u}.$$

Suppose then that $P_t = C$ and that, contrary to our claim, g_{t+1} is not equal to $(1-\delta)g^o(H_{t+1})$. Then, we know that

$$\begin{aligned} C &= (1 - H_{t+1})\overline{\theta} + B\left(\frac{g_{t+1}/(1-\delta)}{(H_{t+1})^{\alpha}}\right) - \left(\frac{c\frac{g_{t+1}}{1-\delta} - \beta c g_{t+1}}{H_{t+1}}\right) + \left(\frac{W_t - \beta W_{t+1}}{H_{t+1}}\right) + \beta C - \underline{u} \\ &< (1 - H_{t+1})\overline{\theta} + S(H_{t+1}) + \left(\frac{W_t - \beta W_{t+1}}{H_{t+1}}\right) + \beta C - \underline{u}. \end{aligned}$$

As an alternative to the policies (W_{t+1}, g_{t+1}), consider the policies $\left(W'_{t+1}, (1-\delta)g^o(H_{t+1})\right)$ where

$$C = (1 - H_{t+1})\overline{\theta} + S(H_{t+1}) + \left(\frac{W_t - \beta W'_{t+1}}{H_{t+1}}\right) + \beta C - \underline{u}.$$

These policies do not change the price P_t. However, since $W'_{t+1} > W_{t+1}$, they relax the intertemporal budget constraint in the sense that

$$\sum_{\tau=t+1}^{\infty} \beta^{\tau-t} \mathcal{W}(H_{\tau+1})(1-\beta) = \beta W_{t+1} < \beta W'_{t+1}.$$

This permits the choice of a preferred sequence of housing levels $\{H'_{\tau+1}\}_{\tau=t}^{\infty}$, which is a contradiction. ■

Fact A.1.5 allows us to eliminate $\{g_{\tau+1}\}_{\tau=t}^{\infty}$ from the choice variables. Moreover, substituting in the optimal public good levels and using the definitions of $\mathcal{W}(H_{\tau+1})$ and W_τ, we can write the period t residents' problem as

$$\max_{\{P_t, W_{t+1}, \{H_{\tau+1}\}_{\tau=t}^{\infty}\}} \left\{ \begin{array}{c} P_t + \mu\overline{\theta} \sum_{\tau=t}^{\infty} (\mu\beta)^{\tau-t} (H_{\tau+1} - 1) + \frac{u}{1-\beta} \\ s.t. \ \frac{H_{t+1}(P_t - C)}{1-\beta} + \sum_{\tau=t}^{\infty} \beta^{\tau-t} \mathcal{W}(H_{\tau+1}) = \frac{W_t}{1-\beta} \\ P_t = C - \frac{(1-\beta)\mathcal{W}(H_{t+1}) - (W_t - \beta W_{t+1})}{H_{t+1}} \\ H_{\tau+1} \geq H_\tau \text{ for all } \tau \\ P_t \leq C \ (= \text{if } H_{t+1} > H_t) \end{array} \right\} \tag{A18}$$

This formulation nicely illustrates how the community is constrained in the population it can attract by its initial wealth W_t and that this constraint can be relaxed at the cost of reducing P_t below C. Reducing P_t below C corresponds to leaving the community with a higher wealth level in period $t+1$ (i.e., increasing W_{t+1}).

With this formulation, we are able to show that all new construction takes place in periods t and $t+1$.

Fact A.1.6. *In the initial residents' optimal plan, for all* $\tau \geq t+3$

$$H_\tau = H_{t+2}.$$

Proof of Fact A.1.6. Suppose the Fact is not true. Let $\tau \geq t+3$ be the first period which violates the claim; that is,

$$H_\tau > H_{\tau-1} = ... = H_{t+2}.$$

Let λ be the multiplier on the intertemporal budget constraint in problem (A18). Given that we can raise $H_{\tau-1}$ marginally without violating any of the constraints, we must have that benefits from so doing, which are $\mu\overline{\theta}(\mu\beta)^{\tau-2-t}$, are no greater than the costs, which are $\lambda\beta^{\tau-2-t}\mathcal{W}'(H_{\tau-1})$. This implies that

$$\mu\overline{\theta}\mu^{\tau-2-t} \leq \lambda\mathcal{W}'(H_{\tau-1}).$$

Given that we can lower H_τ marginally without violating any of the constraints, we must have that the benefits from so doing, which are $\lambda\beta^{\tau-1-t}\mathcal{W}'(H_\tau)$, are less than the costs, which are $\mu\overline{\theta}(\mu\beta)^{\tau-1-t}$. This implies that

$$\lambda\mathcal{W}'(H_\tau) \leq \mu\overline{\theta}\mu^{\tau-1-t}.$$

Combining these two inequalities we have that $\mathcal{W}'(H_\tau) < \mathcal{W}'(H_{\tau-1})$, which contradicts the fact that $\mathcal{W}(H)$ is convex. ■

Fact A.1.6 allows us to eliminate the housing levels $\{H_{\tau+1}\}_{\tau=t+2}^{\infty}$ from the choice variables and write the period t residents' problem as

$$\max_{\{P_t, W_{t+1}, H_{\tau+1}, H_{t+2}\}} \left\{ \begin{array}{c} P_t + \mu\overline{\theta}\left[H_{t+1} - 1 + \frac{\mu\beta}{1-\mu\beta}(H_{t+2} - 1)\right] + \frac{u}{1-\beta} \\ s.t.\ \frac{H_{t+1}(P_t - C)}{1-\beta} + \mathcal{W}(H_{t+1}) + \frac{\beta}{1-\beta}\mathcal{W}(H_{t+2}) = \frac{W_t}{1-\beta} \\ P_t = C - \frac{(1-\beta)\mathcal{W}(H_{t+1}) - (W_t - \beta W_{t+1})}{H_{t+1}} \\ H_{t+2} \geq H_{t+1} \geq H_t \\ P_t \leq C \ (= \text{if } H_{t+1} > H_t) \end{array} \right\}. \tag{A19}$$

Moreover, plugging the market equilibrium constraint into the intertemporal budget constraint reveals that

$$\frac{W_t - \beta W_{t+1}}{1-\beta} + \frac{\beta}{1-\beta}\mathcal{W}(H_{t+2}) = \frac{W_t}{1-\beta},$$

which immediately implies that $H_{t+2} = \mathcal{H}(W_{t+1})$. The period t residents' problem then reduces to

$$\max_{\{W_{t+1}, H_{\tau+1}\}} \left\{ \begin{array}{c} C - \frac{(1-\beta)\mathcal{W}(H_{t+1}) - (W_t - \beta W_{t+1})}{H_{t+1}} + \mu\overline{\theta}\left[H_{t+1} - 1 + \frac{\mu\beta}{1-\mu\beta}(\mathcal{H}(W_{t+1}) - 1)\right] + \frac{u}{1-\beta} \\ s.t.\ \mathcal{H}(W_{t+1}) \geq H_{t+1} \geq H_t \\ W_t - \beta W_{t+1} \leq (1-\beta)\mathcal{W}(H_{t+1}) \ (= \text{if } H_{t+1} > H_t) \end{array} \right\} \tag{A20}$$

This problem involves just a choice of two variables - W_{t+1} and $H_{\tau+1}$ - how much wealth to accumulate in period t and how much new construction to undertake.

The solution to this problem depends on whether H_t is smaller or larger than H^s where H^s is defined in (17). The former situation is the case covered by Proposition 3 and the latter is covered by Proposition 4. We tackle the two cases separately.

The solution when $H_t < H^s$ (Proposition 2)

Fact A.1.7. *Suppose that $H_t < H^s$. Then, in the period t residents' optimal plan, there exist wealth levels $W^*(H_t)$ and $W_n(H_t)$, satisfying $\mathcal{W}(H_t) < W^*(H_t) < W_n(H_t)$, such that*

$$(W_{t+1}, H_{t+1}) = \begin{cases} (W_t, \mathcal{H}(W_t)) & \text{if } W_t \geq W^*(H_t) \\ (W_n(H_t), H_t) & \text{if } W_t < W^*(H_t) \end{cases}.$$

Proof of Fact A.1.7. We know that (W_{t+1}, H_{t+1}) solves problem (A20). There are two possibilities to consider: i) the period t price constraint holds with equality at the optimal policies, and ii) the period t price constraint holds with inequality at the optimal policies. We begin with the first possibility.

Possibility i). If the period t price constraint holds with equality, then $(1-\beta)\mathcal{W}(H_{t+1}) = W_t - \beta W_{t+1}$. It then follows that $H_{t+1} = \mathcal{H}((W_t - \beta W_{t+1})/(1-\beta))$. The constraint that $H_{t+1} \geq H_t$ implies that $\mathcal{H}((W_t - \beta W_{t+1})/(1-\beta)) \geq H_t$ or equivalently that

$$\frac{W_t - (1-\beta)\mathcal{W}(H_t)}{\beta} \geq W_{t+1}.$$

The constraint that $\mathcal{H}(W_{t+1}) \geq H_{t+1}$ implies that $W_{t+1} \geq W_t$. It follows that the range of feasible W_{t+1} values is

$$W_{t+1} \in [W_t, \frac{W_t - (1-\beta)\mathcal{W}(H_t)}{\beta}].$$

For this interval to be non-empty it is necessary that $W_t \geq \mathcal{W}(H_t)$.

Given the price constraint holds with equality, the optimal choice of period $t+1$ wealth must solve the problem

$$\max_{\{W_{t+1}\}} \left\{ \begin{array}{c} \mathcal{H}(\frac{W_t - \beta W_{t+1}}{1-\beta}) + \frac{\mu\beta}{1-\mu\beta}\mathcal{H}(W_{t+1}) \\ s.t.\ W_{t+1} \in [W_t, \frac{W_t-(1-\beta)\mathcal{W}(H_t)}{\beta}] \end{array} \right\}.$$

The derivative of the objective function is

$$\frac{\mu\beta}{1-\mu\beta}\mathcal{H}'(W_{t+1}) - \frac{\beta}{1-\beta}\mathcal{H}'(\frac{W_t - \beta W_{t+1}}{1-\beta}).$$

The concavity of the function $\mathcal{H}(W)$, implies that this derivative is negative for all $W_{t+1} \geq W_t$. The optimal choice of period $t+1$ wealth is therefore W_t. This in turn implies that $H_{t+1} = \mathcal{H}(W_t)$.

We conclude that if the period t price constraint holds with equality at the optimal policies, then the optimal policies are $(W_t, \mathcal{H}(W_t))$. A necessary condition for this to be the solution is that $W_t \geq \mathcal{W}(H_t)$. Note for future reference that the payoff from this candidate solution is

$$C + \frac{\mu\overline{\theta}}{1-\mu\beta}(\mathcal{H}(W_t) - 1) + \frac{\underline{u}}{1-\beta}. \tag{A21}$$

Possibility ii). If the period t price constraint holds as an inequality at the optimal policies, then $(1-\beta)\mathcal{W}(H_{t+1}) > W_t - \beta W_{t+1}$ and $H_{t+1} = H_t$. This means that

$$W_{t+1} > \frac{W_t - (1-\beta)\mathcal{W}(H_t)}{\beta}.$$

The constraint that $\mathcal{H}(W_{t+1}) \geq H_{t+1}$ requires that $W_{t+1} \geq \mathcal{W}(H_t)$. Define the wealth level $W_n(H_t)$ as the solution to the following problem

$$\max_{\{W_{t+1}\}} \left\{ \begin{array}{c} -\frac{\beta W_{t+1}}{H_t} + \mu\overline{\theta}\left(\frac{\mu\beta}{1-\mu\beta}(\mathcal{H}(W_{t+1})-1)\right) \\ s.t.\ W_{t+1} \geq \mathcal{W}(H_t) \end{array} \right\}. \tag{A22}$$

Then, the optimal policies must equal $(W_n(H_t), H_t)$ and it must be the case that

$$W_n(H_t) > \frac{W_t - (1-\beta)\mathcal{W}(H_t)}{\beta}. \tag{A23}$$

The payoff from this candidate solution is

$$C - \left(\frac{(1-\beta)\mathcal{W}(H_t) - (W_t - \beta W_n(H_t))}{H_t}\right) + \mu\overline{\theta}\left((H_t - 1) + \frac{\mu\beta}{1-\mu\beta}(\mathcal{H}(W_n(H_t)) - 1)\right) + \frac{\underline{u}}{1-\beta}. \tag{A24}$$

We now provide some more information about the wealth level $W_n(H_t)$.

Claim A.1.1. *$W_n(H_t) > \mathcal{W}(H_t)$.*

Proof of Claim A.1.1. By definition, the wealth level $W_n(H_t)$ solves problem (A22). The derivative of the objective function in problem (A22) is

$$-\frac{\beta}{H_t} + \mu\overline{\theta}\frac{\mu\beta}{1-\mu\beta}\mathcal{H}'(W_{t+1}),$$

and the second derivative is

$$\mu\overline{\theta}\frac{\mu\beta}{1-\mu\beta}\mathcal{H}''(W_{t+1}) < 0.$$

The objective function is therefore concave. To prove the claim, it suffices to show that if $H_t < H^s$, then it is the case that

$$-\frac{\beta}{H_t} + \mu\overline{\theta}\frac{\mu\beta}{1-\mu\beta}\mathcal{H}'(\mathcal{W}(H_t)) > 0.$$

From (A1), we have that

$$\mathcal{H}'(\mathcal{W}(H_t)) = -\frac{1-\beta}{(1-2H_t)\,\overline{\theta} + S(H_t) + H_t S'(H_t) - C(1-\beta) - \underline{u}}.$$

It follows that

$$\begin{aligned} &-\tfrac{\beta}{H_t} + \mu\overline{\theta}\tfrac{\mu\beta}{1-\mu\beta}\mathcal{H}'(\mathcal{W}(H_t)) \\ &= -\tfrac{\beta}{H_t} + \mu\overline{\theta}\tfrac{\mu\beta}{1-\mu\beta}\left[\tfrac{1-\beta}{C(1-\beta)+\underline{u}+H_t\overline{\theta}-(1-H_t)\overline{\theta}-S(H_t)-H_tS'(H_t)}\right]. \end{aligned} \tag{A25}$$

The right hand side of (A25) is positive if

$$\frac{\mu^2\overline{\theta}(1-\beta)}{1-\mu\beta}\left[\frac{1}{C(1-\beta) + \underline{u} + H_t\overline{\theta} - (1-H_t)\overline{\theta} - S(H_t) - H_t S'(H_t)}\right] > \frac{1}{H_t}.$$

This inequality is equivalent to

$$(1-H_t)\overline{\theta}+S(H_t)+H_tS'(H_t)-C(1-\beta)>\underline{u}+H_t\overline{\theta}\left(1-\frac{\mu^2(1-\beta)}{1-\mu\beta}\right).$$

This follows from the fact that $H_t<H^s$. ■

Finally, note that it follows from the claim and the fact that the wealth level $W_n(H_t)$ solves problem (A22) that it must be the case that

$$\mu\overline{\theta}\frac{\mu}{1-\mu\beta}\mathcal{H}'(W_n(H_t))=\frac{1}{H_t}. \tag{A26}$$

Which possibility arises? Having understood the two possibilities, we can now analyze which one arises. A necessary condition for possibility i) to be the solution is that $W_t\geq\mathcal{W}(H_t)$ and condition (A23) implies that a necessary condition for possibility ii) to be the solution is that $W_t<\beta W_n(H_t)+(1-\beta)\mathcal{W}(H_t)$. Given that $W_n(H_t)>\mathcal{W}(H_t)$, we can conclude that the solution is $(W_n(H_t),H_t)$ if $W_t<\mathcal{W}(H_t)$ and $(W_t,\mathcal{H}(W_t))$ if $W_t\geq\beta W_n(H_t)+(1-\beta)\mathcal{W}(H_t)$. For values of W_t in the interval $[\mathcal{W}(H_t),\beta W_n(H_t)+(1-\beta)\mathcal{W}(H_t))$ both possibilities are feasible. Thus, which possibility is optimal depends on a comparison of the payoffs (A21) and (A24). We can show:

Claim A.1.2. *There exists $W^*(H_t)\in(\mathcal{W}(H_t),\beta W_n(H_t)+(1-\beta)\mathcal{W}(H_t))$ such that the optimal policies are given by:*

$$(W_{t+1},H_{t+1})=\begin{cases}(W_t,\mathcal{H}(W_t)) & \text{if } W_t\geq W^*(H_t)\\ (W_n(H_t),H_t) & \text{if } W_t<W^*(H_t)\end{cases}. \tag{A27}$$

Proof of Claim A.1.2. When $W_t\in[\mathcal{W}(H_t),\beta W_n(H_t)+(1-\beta)\mathcal{W}(H_t)]$, the solution will be $(W_t,\mathcal{H}(W_t))$ if (A21) exceeds (A24) and $(W_n(H_t),H_t)$ if (A21) is less than (A24). Differencing (A21) and (A24) yields

$$\begin{aligned}&\frac{\mu\overline{\theta}}{1-\mu\beta}\mathcal{H}(W_t)+\left(\frac{(1-\beta)\mathcal{W}(H_t)-(W_t-\beta W_n(H_t))}{H_t}\right)-\mu\overline{\theta}\left(H_t+\frac{\mu\beta}{1-\mu\beta}\mathcal{H}(W_n(H_t))\right)\\ =\;&\left(\frac{(1-\beta)\mathcal{W}(H_t)-(W_t-\beta W_n(H_t))}{H_t}\right)-\mu\overline{\theta}\left(H_t+\frac{\mu\beta}{1-\mu\beta}\mathcal{H}(W_n(H_t))-\frac{1}{1-\mu\beta}\mathcal{H}(W_t)\right).\end{aligned}$$

Define the function $\varphi(W;H_t)$ on the interval $[\mathcal{W}(H_t),\beta W_n(H_t)+(1-\beta)\mathcal{W}(H_t)]$ to equal this difference.

Note first that

$$\varphi(\mathcal{W}(H_t);H_t)=\frac{\beta\left(W_n(H_t)-\mathcal{W}(H_t)\right)}{H_t}-\mu\overline{\theta}\left(\frac{\mu\beta}{1-\mu\beta}\left(\mathcal{H}(W_n(H_t))-\mathcal{H}(\mathcal{W}(H_t))\right)\right).$$

By the *Mean-Value Theorem,* there exists $W \in (\mathcal{W}(H_t), W_n(H_t))$ such that

$$\varphi(\mathcal{W}(H_t); H_t) = -\left[\mu\overline{\theta}\frac{\mu}{1-\mu\beta}\mathcal{H}'(W) - \frac{1}{H_t}\right]\beta\left(W_n(H_t) - \mathcal{W}(H_t)\right).$$

The concavity of the function $\mathcal{H}(W)$ then implies that

$$\varphi(\mathcal{W}(H_t); H_t) < -\left[\mu\overline{\theta}\frac{\mu}{1-\mu\beta}\mathcal{H}'(W_n(H_t)) - \frac{1}{H_t}\right]\beta\left(W_n(H_t) - \mathcal{W}(H_t)\right) = 0.$$

On the other hand, we have that

$$\begin{aligned}
& \varphi(\beta W_n(H_t) + (1-\beta)\mathcal{W}(H_t); H_t) \\
= \; & -\mu\overline{\theta}\left(H_t + \frac{\mu\beta}{1-\mu\beta}\mathcal{H}(W_n(H_t)) - \frac{1}{1-\mu\beta}\mathcal{H}(\beta W_n(H_t) + (1-\beta)\mathcal{W}(H_t))\right) \\
= \; & \frac{\mu\overline{\theta}}{1-\mu\beta}\left[\mathcal{H}(\beta W_n(H_t) + (1-\beta)\mathcal{W}(H_t)) - \left((1-\mu\beta)\,H_t + \mu\beta\mathcal{H}(W_n(H_t))\right)\right] \\
> \; & \frac{\mu\overline{\theta}}{1-\mu\beta}\left[\mathcal{H}(\beta W_n(H_t) + (1-\beta)\mathcal{W}(H_t)) - \left((1-\beta)\,\mathcal{H}(\mathcal{W}(H_t)) + \beta\mathcal{H}(W_n(H_t))\right)\right] \\
> \; & 0,
\end{aligned}$$

where the first inequality follows from the fact that $H_t = \mathcal{H}(\mathcal{W}(H_t)) < \mathcal{H}(W_n(H_t))$ and the second inequality follows from the concavity of $\mathcal{H}(W)$.

Finally, we have that for all $W \in [\mathcal{W}(H_t), \beta W_n(H_t) + (1-\beta)\mathcal{W}(H_t)]$

$$\frac{\partial\varphi(W; H_t)}{\partial W} = -\frac{1}{H_t} + \mu\overline{\theta}\left(\frac{1}{1-\mu\beta}\mathcal{H}'(W)\right) > 0,$$

where the inequality follows from the concavity of $\mathcal{H}(W)$ and the fact that $W < W_n(H_t)$.

We conclude that there exists a unique $W^*(H_t) \in (\mathcal{W}(H_t), \beta W_n(H_t) + (1-\beta)\mathcal{W}(H_t))$ such that $\varphi(W_t; H_t) < 0$ if $W_t < W^*(H_t)$ and $\varphi(W_t; H_t) > 0$ if $W_t > W^*(H_t)$. ■

Fact A.1.7 now follows from Claims A.1.1 and A.1.2. ■

We have now completed the characterization of the solution to the period t residents' problem when $H_t < H^s$ and can verify it has the same form as described in Proposition 2. If $W_t \geq W^*(H_t)$, then, given that $H_{t+1} = \mathcal{H}(W_t)$, in period t the housing stock increases to $\mathcal{H}(W_t)$. Moreover, since $g_{t+1} = (1-\delta)g^o(\mathcal{H}(W_t))$, the community invests in $g^o(\mathcal{H}(W_t)) - g_t$ units of the public good in period t. Given that $W_{t+1} = W_t$, it must be the case that

$$c(1-\delta)g^o(\mathcal{H}(W_t)) - (1+\rho)b_{t+1} = cg_t - (1+\rho)b_t,$$

which implies that

$$(1+\rho)b_{t+1} - (1+\rho)b_t = c\left[(1-\delta)g^o(\mathcal{H}(W_t)) - g_t\right].$$

Thus, all but $c\delta g^o(\mathcal{H}(W_t))$ of the cost of investment is financed with debt. Since $H_\tau = \mathcal{H}(W_{t+1})$ and $g_\tau = (1-\delta)g^o(H_\tau)$ for all $\tau \geq t+2$, thereafter, the community maintains the public good at $g^o(\mathcal{H}(W_t))$ and the market provides no more housing. From (A15) we have that $(1+\rho)b_\tau = (1+\rho)b_{t+1}$ for all $\tau \geq t+2$, implying that debt remains constant. This means that the community's wealth remains at W_t and taxes finance the maintenance of the public good and interest on the debt. The price of houses is C in period t and in all subsequent periods.

If $W_t < W^*(H_t)$, then, given that $H_{t+1} = H_t$, no new construction takes place in period t. Moreover, since $g_{t+1} = (1-\delta)g^o(H_t)$, the community invests in $g^o(H_t) - g_t$ units of the public good in period t. Given that $W_{t+1} = W_n(H_t)$, it must be the case that

$$c(1-\delta)g^o(H_t) - (1+\rho)b_{t+1} = W_n(H_t),$$

which implies that

$$(1+\rho)b_{t+1} - (1+\rho)b_t = c\left[(1-\delta)g^o(H_t) - g_t\right] - (W_n(H_t) - W_t)\,.$$

The price of houses in period t is

$$C - \left(\frac{(1-\beta)\mathcal{W}(H_t) - (W_t - \beta W_n(H_t))}{H_t}\right),$$

which is less than C. Given that $H_{t+2} = \mathcal{H}(W_n(H_t))$, in period $t+1$ the housing stock increases to $\mathcal{H}(W_n(H_t))$. Moreover, since $g_{t+2} = (1-\delta)g^o\left(\mathcal{H}(W_n(H_t))\right)$, the community invests in $g^o(\mathcal{H}(W_n(H_t))) - (1-\delta)g^o(H_t)$ units of the public good in period t. From (A15), we have that

$$(1+\rho)b_{t+2} - (1+\rho)b_{t+1} = c\left[(1-\delta)g^o\left(\mathcal{H}(W_n(H_t))\right) - (1-\delta)g^o(H_t)\right],$$

implying that all but $c\delta g^o(\mathcal{H}(W_n(H_t)))$ of the cost of investment is financed with debt. Since $H_\tau = \mathcal{H}(W_n(H_t))$and $g_\tau = (1-\delta)g^o(H_\tau)$ for all $\tau \geq t+3$, thereafter, the community maintains the public good at $g^o(\mathcal{H}(W_n(H_t)))$ and the market provides no more housing. From (A15) we have that $(1+\rho)b_\tau = (1+\rho)b_{t+2}$ for all $\tau \geq t+3$, implying that debt remains constant. This means that the community's wealth remains at $W_n(H_t)$ and taxes finance the maintenance of the public good and interest on the debt. The price of houses is C in period $t+1$ and in all subsequent periods.

To complete the proof of Proposition 3, it only remains to establish that the functions $W^*(H_t)$ and $W_n(H_t)$ defined in Fact A.1.7 have the properties claimed in Proposition 3.

Fact A.1.8. *The functions $W^*(H_t)$ and $W_n(H_t)$ defined in Fact A.1.7 are continuously differentiable and increasing on $[H_0, H^s)$, and satisfy the limit condition that $\lim_{H_t \nearrow H^s} W^*(H_t) = \lim_{H_t \nearrow H^s} W_n(H_t) = \mathcal{W}(H^s)$.*

Proof of Fact A.1.8. We begin with the function $W_n(H_t)$. From the proof of Fact A.1.7, we know that $W_n(H_t)$ satisfies (A26). It follows immediately from the concavity of $\mathcal{H}(W)$ that $W_n(H_t)$ is increasing in H_t. The definition of H^s implies that

$$\mu\overline{\theta}\frac{\mu}{1-\mu\beta}\mathcal{H}'(\mathcal{W}(H^s)) = \frac{1}{H^s}.$$

Along with (A26), this implies that $\lim_{H_t \nearrow H^s} W_n(H_t) = \mathcal{W}(H^s)$.

We now turn to the function $W^*(H_t)$. We know that $W^*(H_t) \in (\mathcal{W}(H_t), \beta W_n(H_t) + (1-\beta)\mathcal{W}(H_t))$. We also claim that $W^*(H_t)$ is increasing. To see this, note that $W^*(H_t)$ is implicitly defined by the equation $\varphi(W^*(H_t); H_t) = 0$. It follows that

$$\frac{dW^*}{dH_t} = -\frac{\frac{\partial\varphi(W^*;H_t)}{\partial H_t}}{\frac{\partial\varphi(W^*;H_t)}{\partial W}}.$$

We have already established that $\frac{\partial\varphi(W^*;H_t)}{\partial W} > 0$. Thus, to establish the claim we need to show that $\frac{\partial\varphi(W^*;H_t)}{\partial H_t} < 0$. Differentiating and using the first order condition (A26), we have that

$$\begin{aligned}
\frac{\partial\varphi(W^*;H_t)}{\partial H_t} &= \frac{(1-\beta)\mathcal{W}'(H_t)}{H_t} - \frac{[(1-\beta)\mathcal{W}(H_t) - (W^* - \beta W_n(H_t))]}{(H_t)^2} - \mu\overline{\theta} \\
&< \frac{(1-\beta)\mathcal{W}'(H_t)}{H_t} - \mu\overline{\theta},
\end{aligned}$$

where the inequality follows from the fact that

$$\frac{[(1-\beta)\mathcal{W}(H_t) - (W^* - \beta W_n(H_t))]}{(H_t)^2} > 0.$$

Using (A2) and the definition of $\mathcal{W}(H_t)$, we have that

$$\frac{(1-\beta)\mathcal{W}'(H_t)}{H_t} = \frac{(1-\beta)\mathcal{W}(H_t)}{(H_t)^2} + \overline{\theta} - S'(H_t).$$

Thus,

$$\frac{\partial\varphi(W^*;H_t)}{\partial H_t} < \frac{(1-\beta)\mathcal{W}(H_t)}{(H_t)^2} + \overline{\theta}(1-\mu) - S'(H_t).$$

It therefore suffices to show that

$$\frac{(1-\beta)\mathcal{W}(H_t)}{H_t} + H_t\overline{\theta}(1-\mu) - H_t S'(H_t) < 0.$$

Using the definition of $\mathcal{W}(H_t)$, we have that

$$\begin{aligned}
&\frac{(1-\beta)\mathcal{W}(H_t)}{H_t} + H_t\overline{\theta}(1-\mu) - H_t S'(H_t) \\
= \;& H_t\overline{\theta}(1-\mu) + C(1-\beta) + \underline{u} - (1-H_t)\overline{\theta} - S(H_t) - H_t S'(H_t).
\end{aligned}$$

But since $H_t < H^s$, we have that

$$\begin{aligned}(1-H_t)\overline{\theta} - S\left(H_t\right) - H_t S'\left(H_t\right) &> C(1-\beta) + \underline{u} + H_t\overline{\theta}\left(1 - \frac{\mu^2(1-\beta)}{1-\mu\beta}\right) \\ &> C(1-\beta) + \underline{u} + H_t\overline{\theta}\left(1-\mu\right),\end{aligned}$$

where the last inequality follows from the fact that

$$\frac{\mu^2(1-\beta)}{1-\mu\beta} < \mu.$$

Given that $W^*(H_t) \in (\mathcal{W}(H_t), \beta W_n(H_t) + (1-\beta)\mathcal{W}(H_t))$ and that $\lim_{H_t \nearrow H^s} W_n(H_t) = \mathcal{W}(H^s)$, it must also be the case that $\lim_{H_t \nearrow H^s} W^*(H_t) = \mathcal{W}(H^s)$. ■

The solution when $H_t \geq H^s$ (Proposition 3)

Fact A.1.9. *Suppose that $H_t \geq H^s$. Then, in the period t residents' optimal plan,*

$$(W_{t+1}, H_{t+1}) = \begin{cases} (W_t, \mathcal{H}\left(W_t\right)) & \text{if } W_t > \mathcal{W}(H_t) \\ (\mathcal{W}(H_t), H_t) & \text{if } W_t \leq \mathcal{W}(H_t) \end{cases}.$$

Proof of Fact A.1.9. We know that (W_{t+1}, H_{t+1}) solves problem (A20). There are two possibilities to consider: i) the period t price constraint holds with equality at the optimal policies, and ii) the period t price constraint holds with inequality at the optimal policies. We begin with the first possibility.

Possibility i). As argued in the proof of Fact A.1.7, in this case the optimal choice of period $t+1$ wealth is W_t which implies that $H_{t+1} = \mathcal{H}(W_t)$. Furthermore, a necessary condition for this to be the solution is that $W_t \geq \mathcal{W}(H_t)$.

Possibility ii). As argued in the proof of Fact A.1.7, the optimal W_{t+1} must solve the problem

$$\max_{\{W_{t+1}\}} \left\{ \begin{array}{c} -\frac{\beta W_{t+1}}{H_t} + \mu\overline{\theta}\left(\frac{\mu\beta}{1-\mu\beta}(\mathcal{H}(W_{t+1})-1)\right) \\ s.t.\ W_{t+1} \geq \mathcal{W}(H_t) \end{array} \right\}. \tag{A28}$$

We claim that the solution is $W_{t+1} = \mathcal{W}(H_t)$. To show this, it is enough to show that

$$\mu\overline{\theta}\frac{\mu\beta}{1-\mu\beta}\mathcal{H}'(\mathcal{W}(H_t)) \leq \frac{\beta}{H_t}.$$

Following the steps in the proof of Fact A.1.7, this inequality is equivalent to

$$(1-H_t)\overline{\theta} + S\left(H_t\right) + H_t S'\left(H_t\right) - C(1-\beta) \leq \underline{u} + H_t\overline{\theta}\left(1 - \frac{\mu^2(1-\beta)}{1-\mu\beta}\right).$$

This follows from the fact that $H_t \geq H^s$. Given that $W_{t+1} = \mathcal{W}(H_t)$, for the price constraint to hold as an inequality it must be the case that

$$\mathcal{W}(H_t) > W_t. \tag{A29}$$

Given that a necessary condition for possibility i) is that $W_t \geq \mathcal{W}(H_t)$ and a necessary condition for possibility ii) is that $W_t < \mathcal{W}(H_t)$, the result follows immediately. ■

We have now completed the characterization of the solution to the period t residents' problem when $H_t \geq H^s$ and can verify it is as described in Proposition 4. If $W_t \geq W^*(H_t)$, then, given that $H_{t+1} = \mathcal{H}(W_t)$, in period t the housing stock increases to $\mathcal{H}(W_t)$. Moreover, since $g_{t+1} = (1-\delta)g^o(\mathcal{H}(W_t))$, the community invests in $g^o(\mathcal{H}(W_t)) - g_t$ units of the public good in period t. Given that $W_{t+1} = W_t$, it must be the case that

$$c(1-\delta)g^o(\mathcal{H}(W_t)) - (1+\rho)b_{t+1} = cg_t - (1+\rho)b_t,$$

which implies that

$$(1+\rho)b_{t+1} - (1+\rho)b_t = c\left[(1-\delta)g^o(\mathcal{H}(W_t)) - g_t\right].$$

Thus, all but $c\delta g^o(\mathcal{H}(W_t))$ of the cost of investment is financed with debt. Since $H_\tau = \mathcal{H}(W_{t+1})$ and $g_\tau = (1-\delta)g^o(H_\tau)$ for all $\tau \geq t+2$, thereafter, the community maintains the public good at $g^o(\mathcal{H}(W_t))$ and the market provides no more housing. From (A15) we have that $(1+\rho)b_\tau = (1+\rho)b_{t+1}$ for all $\tau \geq t+2$, implying that debt remains constant. This means that the community's wealth remains at W_t and taxes finance the maintenance of the public good and interest on the debt. The price of houses is C in period t and in all subsequent periods.

If $W_t < W^*(H_t)$, then, given that $H_{t+1} = H_t$, no new construction takes place in period t. Moreover, since $g_{t+1} = (1-\delta)g^o(H_t)$, the community invests in $g^o(H_t) - g_t$ units of the public good in period t. Given that $W_{t+1} = \mathcal{W}(H_t)$, it must be the case that

$$c(1-\delta)g^o(H_t) - (1+\rho)b_{t+1} = \mathcal{W}(H_t),$$

which implies that

$$(1+\rho)b_{t+1} - (1+\rho)b_t = c\left[(1-\delta)g^o(H_t) - g_t\right] - (\mathcal{W}(H_t) - W_t).$$

The price of houses in period t is

$$C - \left(\frac{\mathcal{W}(H_t) - W_t}{H_t}\right),$$

which is less than C. Given that $H_{t+2} = \mathcal{H}(\mathcal{W}(H_t)) = H_t$, no new construction takes place in period $t+1$. Moreover, since $g_{t+2} = (1-\delta)g^o\left(\mathcal{H}(\mathcal{W}(H_t))\right)$, the community maintains the public good level at $g^o(H_t)$ in period $t+1$. From (A15), we have that

$$(1+\rho)b_{t+2} - (1+\rho)b_{t+1} = 0,$$

so there is no change in debt. Since $H_\tau = \mathcal{H}(\mathcal{W}(H_t))$and $g_\tau = (1-\delta)g^o(H_\tau)$ for all $\tau \geq t+3$, thereafter, the community maintains the public good at $g^o(H_t)$ and the market provides no more housing. From (A15) we have that $(1+\rho)b_\tau = (1+\rho)b_{t+2}$ for all $\tau \geq t+3$, implying that debt remains constant. This means that the community's wealth remains at $\mathcal{W}(H_t)$ and taxes finance the maintenance of the public good and interest on the debt. The price of houses is C in period $t+1$ and in all subsequent periods.

It should be stressed here that increasing wealth from W_t to $\mathcal{W}(H_t)$ is not the only way of financing the public good. It would be equally good to just keep wealth constant at W_t. The key point is that as long as the population remains constant at H_t, which requires that wealth is does not exceed $\mathcal{W}(H_t)$, a form of Ricardian equivalence holds. Specifically, tax and debt finance are equivalent. Intuitively, if the residents increase tax finance and reduce debt finance, if they remain in the community, they will benefit from the lower taxes its higher wealth allows and if they leave, the higher wealth will be capitalized into the price of housing. Our method of solution picks out a financing patth in which wealth increases to $\mathcal{W}(H_t)$ because of Fact A.1.4. Recall, that this established that there is no loss of generality in assuming that future housing prices are equal to C. This was then used to restrict attention to paths in which this was the case. ■

Appendix 2: Proof of Theorem

Let $W^* \in \Psi$ and let $\xi(W^*)$ be the associated candidate equilibrium. We need to show that $\xi(W^*)$ is an equilibrium if W^* satisfies the conditions of the Theorem. This requires showing that the policy rules and value function introduced in Section 5.3 satisfy the conditions for equilibrium described in Section 3.3. Recall that there are two such conditions. The first is that, for all states (g, b, H), the policy rules solve the residents' problem (8) when the continuation value is described by the value function and next period's price of housing is determined by the equilibrium price rule evaluated at next period's state variables. The second is that, for all states (g, b, H), the value function satisfies the equality (9).

It is helpful to write out problem (8) with all its constraints as follows:

$$\max_{(g',b',T,H',P)} \left\{ \begin{array}{c} (1-\mu)\left[P + \frac{u}{1-\beta}\right] + \mu\left[B\left(\frac{g'/(1-\delta)}{(H')^{\alpha}}\right) - T + \beta V(g', b', H')\right] \\ s.t.\ (1+\rho)b + c\left(\frac{g'}{1-\delta} - g\right) = b' + H'T \\ P = (1-H')\overline{\theta} + B(\frac{g'/(1-\delta)}{(H')^{\alpha}}) - T + \beta P(g', b', H') - \underline{u} \\ H' \geq H \\ P \leq C\ (= \text{if } H' > H). \end{array} \right\}.$$

Solving the budget constraint for the tax T, substituting this into the objective function and market equilibrium condition, and using the notation W and W' to describe current and future wealth, we can remove the tax as a choice variable and write the problem as:

$$\max_{(g',W',H',P)} \left\{ \begin{array}{c} (1-\mu)\left[P + \frac{u}{1-\beta}\right] \\ +\mu\left[B(\frac{g'/(1-\delta)}{(H')^{\alpha}}) - \frac{c\left(\frac{g'}{1-\delta} - \beta g'\right)}{H'} + \frac{W - \beta W'}{H'} + \beta V(g', \beta(W' - cg'), H')\right] \\ s.t.\quad P = (1-H')\overline{\theta} + B(\frac{g'/(1-\delta)}{(H')^{\alpha}}) - B(\frac{g'/(1-\delta)}{(H')^{\alpha}}) - \frac{c\left(\frac{g'}{1-\delta} - \beta g'\right)}{H'} \\ + \frac{W - \beta W'}{H'} + \beta P(g', \beta(W' - cg'), H') - \underline{u} \\ H' \geq H \\ P \leq C\ (= \text{if } H' > H). \end{array} \right\} \tag{A30}$$

While the wealth level W' replaces the debt level b' in the set of choice variables, the latter can be immediately recovered from the equation $b' = \beta(W' - cg')$. This is the form of the residents' problem we will work with.

Since behavior in our candidate equilibrium differs in states in which the housing stock is greater than H^s and less than H^s, we tackle the two situations separately. We begin with the former case, since it is simpler. This is not only because the policy rules and value function are simpler in this range of the state space but also because the durability of housing means that once the community is in this part of the state space it must remain in it.

States (g, b, H) such that $H \geq H^s$

The residents' problem

We need to show that the policy rules defined by (19), (21), (22), and (24) solve the residents' problem (A30) given that the future price and continuation value are as described in (24) and (25) and (26). Our first observation is that we can assume that the residents' policy choices are such that next period's wealth is at least as big as the threshold $\mathcal{W}(H')$.

Fact A.2.1. *Suppose that $H \geq H^s$ and let (g', W', H', P) solve problem (A30) with future price and continuation value as described by (24) and (25) and (26). Then, we may assume without loss of generality that $W' \geq \mathcal{W}(H')$.*

Proof of Fact A.2.1. Suppose that $W' < \mathcal{W}(H')$. We will show that increasing W' to $\mathcal{W}(H')$ will not violate the constraints or change the value of the objective function in problem (A30). Regarding the former, note that using the equilibrium price rule (24), the definition of $\mathcal{P}$ in (23), and the definition of $\mathcal{W}(H')$ in (16), we have

$$\begin{aligned}
P &= (1-H')\overline{\theta} + B(\frac{g'/(1-\delta)}{(H')^{\alpha}}) - \frac{c\left(\frac{g'}{1-\delta} - \beta g'\right)}{H'} + \frac{W - \beta W'}{H'} + \beta P(g', \beta\left(W' - cg'\right), H') - \underline{u} \\
&= (1-H')\overline{\theta} + B(\frac{g'/(1-\delta)}{(H')^{\alpha}}) - \frac{c\left(\frac{g'}{1-\delta} - \beta g'\right)}{H'} + \frac{W - \beta W'}{H'} + \beta \mathcal{P}(H', \mathcal{W}(H'), W') - \underline{u} \\
&= (1-H')\overline{\theta} + B(\frac{g'/(1-\delta)}{(H')^{\alpha}}) - \frac{c\left(\frac{g'}{1-\delta} - \beta g'\right)}{H'} + \frac{W - \beta \mathcal{W}(H')}{H'} + \beta \mathcal{P}(H', \mathcal{W}(H'), \mathcal{W}(H')) - \underline{u} \\
&= (1-H')\overline{\theta} + B(\frac{g'/(1-\delta)}{(H')^{\alpha}}) - \frac{c\left(\frac{g'}{1-\delta} - \beta g'\right)}{H'} + \frac{W - \beta \mathcal{W}(H')}{H'} + \beta C - \underline{u}.
\end{aligned}$$

Regarding the latter, note that using the equilibrium value function (25) and the definition of $\mathcal{W}(H')$ in (16), we have

$$
\begin{aligned}
& B(\frac{g'/(1-\delta)}{(H')^{\alpha}}) - \frac{c\left(\frac{g'}{1-\delta} - \beta g'\right)}{H'} + \frac{W - \beta W'}{H'} + \beta V(g', \beta\left(W' - cg'\right), H') \\
= \; & B(\frac{g'/(1-\delta)}{(H')^{\alpha}}) - \frac{c\left(\frac{g'}{1-\delta} - \beta g'\right)}{H'} + \frac{W - \beta W'}{H'} + \beta\left[V^*(\mathcal{W}(H')) + \frac{W' - \mathcal{W}(H')}{H'}\right] \\
= \; & B(\frac{g'/(1-\delta)}{(H')^{\alpha}}) - \frac{c\left(\frac{g'}{1-\delta} - \beta g'\right)}{H'} + \frac{W - \beta \mathcal{W}(H')}{H'} + \beta V^*(\mathcal{W}(H')).
\end{aligned}
$$

■

Fact A.2.1 simplifies matters considerably as it ties down the form of the value function and fixes the future housing price at C. It allows us to rewrite problem (A30) as

$$
\max_{(g',W',H',P)} \left\{ \begin{array}{c} (1-\mu)\left[P + \frac{\underline{u}}{1-\beta}\right] + \mu\left[B\left(\frac{g'/(1-\delta)}{(H')^{\alpha}}\right) - \frac{c\left(\frac{g'}{1-\delta} - \beta g'\right)}{H'} + \frac{W-\beta W'}{H'} + \beta V^*(W')\right] \\ s.t.\ P = (1-H')\overline{\theta} + B(\frac{g'/(1-\delta)}{(H')^{\alpha}}) - \frac{c\left(\frac{g'}{1-\delta} - \beta g'\right)}{H'} + \frac{W-\beta W'}{H'} + \beta C - \underline{u} \\ H' \geq H \\ P \leq C \ (= \text{if } H' > H) \\ W' \geq \mathcal{W}(H') \end{array} \right\}. \tag{A31}
$$

Our next result shows that public good provision is efficient.

Fact A.2.2. *Suppose that $H \geq H^s$ and let (g', W', H', P) solve problem (A31) with $V^*(W')$ given by (26). Then, $g' = (1-\delta)g^o(H')$.*

Proof of Fact A.2.2. Note first that

$$
(1-\delta)g^o(H') = \arg\max_{g'} \left\{ B\left(\frac{g'/(1-\delta)}{(H')^{\alpha}}\right) - \frac{c\left(\frac{g'}{1-\delta} - \beta g'\right)}{H'} \right\}.
$$

Thus, provided that such a choice does not make P greater than C, it will clearly be optimal to set g' equal to $(1-\delta)g^o(H')$. Suppose then that such a choice does violate the price constraint; i.e.,

$$
(1-H')\overline{\theta} + S(H') + \frac{W - \beta W'}{H'} + \beta C - \underline{u} > C.
$$

Then clearly it must be the case that the price constraint binds under the policies (g', W', H'), which means that

$$
C = (1-H')\overline{\theta} + B(\frac{g'/(1-\delta)}{(H')^{\alpha}}) - \frac{c\left(\frac{g'}{1-\delta} - \beta g'\right)}{H'} + \frac{W - \beta W'}{H'} + \beta C - \underline{u}.
$$

This implies that the payoff from the policies (g', W', H') is

$$(1-\mu\beta)\left(C+\frac{\underline{u}}{1-\beta}\right)+\mu\overline{\theta}(H'-1)+\mu\beta V^*(W').$$

Now choose $\widehat{W}$ greater than W' to satisfy the price constraint with the efficient level of the public good; i.e.,

$$(1-H')\overline{\theta}+S(H')+\frac{W-\beta\widehat{W}}{H'}+\beta C-\underline{u}=C.$$

Consider the alternative policies $((1-\delta)g^o(H'), \widehat{W}, H')$ which involve the efficient level of public good and a higher level of wealth passed to next period's residents. Clearly, these policies satisfy the constraints. Moreover, the payoff from them is

$$(1-\mu\beta)\left(C+\frac{\underline{u}}{1-\beta}\right)+\mu\overline{\theta}(H'-1)+\mu\beta V^*(\widehat{W}).$$

This exceeds the payoff from the policies (g', W', H') since $V^*(\cdot)$ is increasing in wealth. ■

Fact A.2.2 allows us to write the residents' problem as

$$\max_{(W',H',P)}\left\{\begin{array}{c}(1-\mu)\left[P+\frac{\underline{u}}{1-\beta}\right]+\mu\left[S(H')+\frac{W-\beta W'}{H'}+\beta V^*(W')\right]\\ s.t.\ P=(1-H')\overline{\theta}+S(H')+\frac{W-\beta W'}{H'}+\beta C-\underline{u}\\ H'\geq H\\ P\leq C\ (=\text{if } H'>H)\\ W'\geq\mathcal{W}(H')\end{array}\right\}$$

or, eliminating P, as

$$\max_{(W',H')}\left\{\begin{array}{c}(1-\mu)\left[(1-H')\overline{\theta}+\beta C-\underline{u}+\frac{\underline{u}}{1-\beta}\right]+S(H')+\frac{W-\beta W'}{H'}+\mu\beta V^*(W')\\ s.t.\ H'\geq H\\ (1-H')\overline{\theta}+S(H')+\frac{W-\beta W'}{H'}-(1-\beta)C\leq\underline{u}\ (=\text{if } H'>H)\\ W'\geq\mathcal{W}(H')\end{array}\right\} \tag{A32}$$

Our next result ties down the optimal (W', H').

Fact A.2.3. *Suppose that $H\geq H^s$ and let (W', H') solve problem (A32) with $V^*(W')$ given by (26). Then,*

$$(W',H')=\left\{\begin{array}{ll}(\mathcal{W}(H),H) & \text{if } W\leq\mathcal{W}(H)\\ (W,\mathcal{H}(W)) & \text{if } W\in(\mathcal{W}(H),\mathcal{W}(1)]\end{array}\right..$$

Proof of Fact A.2.3. There are two possibilities to consider: i) the price constraint holds with equality at the optimal policies, and ii) the price constraint holds with inequality at the optimal policies. We begin with the first possibility.

Possibility i). If the price constraint holds with equality, then the definition of $\mathcal{W}(H')$ implies that $(1-\beta)\mathcal{W}(H') = W - \beta W'$. It then follows that $H' = \mathcal{H}((W - \beta W')/(1-\beta))$. The constraint that $H' \geq H$ implies that $\mathcal{H}((W - \beta W')/(1-\beta)) \geq H$ or equivalently that

$$\frac{W - (1-\beta)\mathcal{W}(H)}{\beta} \geq W'.$$

The constraint that $W' \geq \mathcal{W}(H')$ implies that $\mathcal{H}(W') \geq H' = \mathcal{H}((W - \beta W')/(1-\beta))$ and hence that $W' \geq W$. It follows that the range of feasible W' values is

$$W' \in [W, \frac{W - (1-\beta)\mathcal{W}(H)}{\beta}].$$

For this interval to be non-empty it is necessary that $W \geq \mathcal{W}(H)$.

Using the fact that the price constraint holds with equality in problem (A32) together with (26), the optimal choice of wealth must solve the problem

$$\max_{\{W'\}} \left\{ \begin{array}{c} \mathcal{H}(\frac{W-\beta W'}{1-\beta}) + \frac{\mu\beta}{1-\mu\beta}\mathcal{H}(W') \\ s.t.\ W' \in [W, \frac{W-(1-\beta)\mathcal{W}(H)}{\beta}] \end{array} \right\}.$$

The derivative of the objective function in this problem is

$$\frac{\mu\beta}{1-\mu\beta}\mathcal{H}'(W') - \frac{\beta}{1-\beta}\mathcal{H}'(\frac{W-\beta W'}{1-\beta}).$$

The concavity of the function $\mathcal{H}(W)$, implies that this derivative is negative for all $W' \geq W$. The optimal choice of wealth is therefore W. This in turn implies that $H' = \mathcal{H}(W)$.

We conclude that if the price constraint holds with equality at the optimal policies, then the optimal policies are $(W, \mathcal{H}(W))$. A necessary condition for this to be the solution is that $W \geq \mathcal{W}(H)$.

Possibility ii). If the price constraint holds as an inequality at the optimal policies, then $(1-\beta)\mathcal{W}(H') > W - \beta W'$ and $H' = H$. This means that

$$W' > \frac{W - (1-\beta)\mathcal{W}(H)}{\beta}.$$

The constraint that $W' \geq \mathcal{W}(H')$ requires that $W' \geq \mathcal{W}(H)$. From (A32) and (26), the optimal choice of wealth must solve the problem

$$\max_{\{W'\}} \left\{ \begin{array}{c} -\frac{\beta W'}{H} + \mu\overline{\theta}\left(\frac{\mu\beta}{1-\mu\beta}(\mathcal{H}(W')-1)\right) \\ s.t.\ W' \geq \mathcal{W}(H) \end{array} \right\}.$$

We claim that the solution is $W' = \mathcal{W}(H)$. To show this, it is enough to show that

$$\mu\overline{\theta}\frac{\mu\beta}{1-\mu\beta}\mathcal{H}'(\mathcal{W}(H)) \leq \frac{\beta}{H}.$$

Following the steps in the proof of Fact A.1.7, this inequality is equivalent to

$$(1-H)\overline{\theta} + S(H) + HS'(H) - C(1-\beta) \leq \underline{u} + H\overline{\theta}\left(1 - \frac{\mu^2(1-\beta)}{1-\mu\beta}\right).$$

This follows from the fact that $H \geq H^s$. Given that $W' = \mathcal{W}(H)$, for the price constraint to hold as an inequality it must be the case that

$$\mathcal{W}(H) > \frac{W - (1-\beta)\mathcal{W}(H)}{\beta},$$

which requires that $W < \mathcal{W}(H)$.

We conclude that if the price constraint holds as an inequality at the optimal policies, then the optimal policies are $(\mathcal{W}(H), H)$. A necessary condition for this to be the solution is that $W < \mathcal{W}(H)$.

Which possibility arises? Having understood the two possibilities, we can now analyze which one arises. A necessary condition for possibility ii) to be the solution is that $W < \mathcal{W}(H)$. Furthermore, a necessary condition for possibility i) to be the solution is that $W \geq \mathcal{W}(H)$. Thus, we conclude that the optimal policies are as described in the statement of the fact. ■

Using Facts A.2.2 and A.2.3, it is clear that the residents want to follow the equilibrium policy rules described in (19), (21), and (22). To verify the price rule, note that

$$P(g,b,H) = \begin{cases} (1-H)\overline{\theta} + S(H) + \frac{W-\beta\mathcal{W}(H)'}{H} + \beta C - \underline{u} & \text{if } W \leq \mathcal{W}(H) \\ C & \text{if } W \in (\mathcal{W}(H), \mathcal{W}(1)] \end{cases}.$$

Given (23), this is equal to (24).

Verifying form of value function

It remains to verify that the value function as described by (25) and (26) satisfies (9). Suppose first that (g, b, H) is such that $W \in (\mathcal{W}(H), \mathcal{W}(1)]$. Then, we have that

$$\begin{aligned}
& (1-\mu)\left[P(\cdot)+\frac{u}{1-\beta}\right]+\mu\left[B\left(\frac{g'(\cdot)/(1-\delta)}{H'(\cdot)^{\alpha}}\right)-T(\cdot)+\beta V(g'(\cdot),b'(\cdot),H'(\cdot))\right] \\
= \; & (1-\mu)\left[C+\frac{u}{1-\beta}\right]+\mu\left[S(\mathcal{H}(W))+\frac{(1-\beta)W}{\mathcal{H}(W)}+\beta V^*(W)\right] \\
= \; & (1-\mu)\left[C+\frac{u}{1-\beta}\right]+\mu\left[\underline{u}+(1-\beta)C+\overline{\theta}\left(\mathcal{H}(W)-1\right)+\beta V^*(W)\right] \\
= \; & (1-\mu)\left[C+\frac{u}{1-\beta}\right]+\mu\left[\underline{u}+(1-\beta)C+\overline{\theta}\left(\mathcal{H}(W)-1\right)\right]+\mu\beta\left[C+\frac{u}{1-\beta}+\frac{\mu\overline{\theta}}{1-\mu\beta}\left(\mathcal{H}(W)-1\right)\right] \\
= \; & V^*(W),
\end{aligned}$$

as required. Now suppose that (g, b, H) is such that $W < \mathcal{W}(H)$. Then, we have that

$$\begin{aligned}
& (1-\mu)\left[P(\cdot)+\frac{u}{1-\beta}\right]+\mu\left[B\left(\frac{g'(\cdot)/(1-\delta)}{H'(\cdot)^{\alpha}}\right)-T(\cdot)+\beta V(g'(\cdot),b'(\cdot),H'(\cdot))\right] \\
= \; & (1-\mu)\left[\mathcal{P}(H,\mathcal{W}(H),W)+\frac{u}{1-\beta}\right]+\mu\left[S(H)+\frac{W-\beta\mathcal{W}(H)}{H}+\beta V^*(\mathcal{W}(H))\right] \\
= \; & (1-\mu)\left[C+\frac{u}{1-\beta}\right]+\mu\left[S(H)+\frac{(1-\beta)\mathcal{W}(H)}{H}+\beta V^*(\mathcal{W}(H))\right]+\frac{W-\mathcal{W}(H)}{H} \\
= \; & (1-\mu)\left[C+\frac{u}{1-\beta}\right]+\mu\left[\underline{u}+(1-\beta)C+\overline{\theta}\left(H-1\right)\right]+ \\
& \mu\beta\left[C+\frac{u}{1-\beta}+\frac{\mu\overline{\theta}}{1-\mu\beta}\left(H-1\right)\right]+\frac{W-\mathcal{W}(H)}{H} \\
= \; & V^*(\mathcal{W}(H))+\frac{W-\mathcal{W}(H)}{H},
\end{aligned}$$

as required.

States (g, b, H) such that $H < H^s$

This case is more complicated both because the policy rules and value function are more complicated in this part of the state space and also because the residents' choices could in principle take the community out of this range of the state space (i.e., by choosing $H \geq H^s$).

Preliminaries

The policy rules and value function depend on the functions $H_c(W)$ and $W_n(H)$ which are defined by (30) and (35). We begin by presenting two key results that establish some important properties of these functions. The first result concerns the function $H_c(W)$.

Fact A.2.4 *If $W \in [W^*(H), \mathcal{W}(H^s))$ and $H < H^s$, then $H_c(W)$ is uniquely defined, belongs to the interval (H, H^s), and is increasing in W. Moreover, for any starting housing level $H < H^s$, the sequence $\langle H_t \rangle_{t=1}^{\infty}$ defined inductively by $H_1 = H_c(W^*(H))$ and $H_t = H_c(W^*(H_{t-1}))$ converges monotonically to H^s. Similarly, for any starting wealth level $W \in [W^*(H_0), \mathcal{W}(H^s))$, the sequence $\langle W_t \rangle_{t=1}^{\infty}$ defined inductively by $W_1 = W^*(H_c(W))$ and $W_t = W^*(H_c(W_{t-1}))$ converges monotonically to $\mathcal{W}(H^s)$.*

Proof of Fact A.2.4 Using (23) and (30), $H_c(W)$ satisfies the equation $\mathcal{P}(H_c, W^*(H_c), W) = C$. Because $W \in [W^*(H), \mathcal{W}(H^s))$ and $W^* \in \Psi$ we know from the assumed properties of $W^*(H)$ that

$$\mathcal{P}(H, W^*(H), W) \geq \mathcal{P}(H, W^*(H), W^*(H)) > C,$$

and that

$$\mathcal{P}(H^s, W^*(H^s), W) < \mathcal{P}(H^s, W^*(H^s), W^*(H^s)) = C.$$

Thus, by the *Intermediate Value Theorem*, there must exist a solution $H_c \in (H, H^s)$ at which $\mathcal{P}(H_c, W^*(H_c), W) = C$. Moreover, at this solution, we must have that $W^*(H_c) > W$. If not, then $W^*(H_c) \leq W$. But then we have that

$$C = \mathcal{P}(H_c, W^*(H_c), W) \geq \mathcal{P}(H_c, W^*(H_c), W^*(H_c)) > C,$$

which is a contradiction.

For uniqueness, it is sufficient that

$$\frac{d\mathcal{P}(H, W^*(H), W)}{dH} < 0$$

at any solution of the equation $\mathcal{P}(H, W^*(H), W) = C$. Note from (23), that

$$\frac{d\mathcal{P}(H, W^*(H), W)}{dH} = -\overline{\theta} + S'(H) - \frac{\beta}{H}\frac{dW^*(H)}{dH} - \frac{(W - \beta W^*(H))}{H^2}.$$

Moreover, at a solution

$$-\frac{(W - \beta W^*(H))}{H} = \left[(1 - H)\overline{\theta} + S(H) - C(1 - \beta) - \underline{u}\right].$$

Thus, at a solution

$$\begin{aligned}
\frac{d\mathcal{P}(H, W^*(H), W)}{dH} &= -\overline{\theta} + S'(H) - \frac{\beta}{H}\frac{dW^*(H)}{dH} + \frac{\left[(1 - H)\overline{\theta} + S(H) - C(1 - \beta) - \underline{u}\right]}{H} \\
&= -\left[\frac{C(1 - \beta) + \underline{u} - (1 - 2H)\overline{\theta} - HS'(H) - S(H) + \beta\frac{dW^*(H)}{dH}}{H}\right].
\end{aligned}$$

Given that $W^*(H)$ is increasing, we have

$$\frac{d\mathcal{P}(H, W^*(H), W)}{dH} < -\left[\frac{\underline{u} + H\overline{\theta} - \left((1-H)\overline{\theta} + S(H) + HS'(H) - C(1-\beta)\right)}{H}\right] < 0$$

where the last inequality follows from Assumptions 1(i) and 2.

A similar logic implies that $H_c(W)$ is increasing in W. Given that the solution satisfies $C = \mathcal{P}(H_c, W^*(H_c), W)$, we have that

$$\frac{dH_c}{dW} = -\frac{\frac{\partial \mathcal{P}(H_c, W^*(H_c), W)}{\partial W}}{\frac{d\mathcal{P}(H_c, W^*(H_c), W)}{dH}} = -\frac{1}{H_c \frac{d\mathcal{P}(H_c, W^*(H_c), W)}{dH}}.$$

Given that $d\mathcal{P}(H_c, W^*(H_c), W)/dH$ is negative, the result follows.

Now let $H < H^s$ and consider the sequence $\langle H_t \rangle_{t=0}^{\infty}$ defined inductively by $H_1 = H_c(W^*(H))$ and $H_t = H_c(W^*(H_{t-1}))$. We first show the sequence is increasing. To see this, recall that we showed that for any $H < H^s$, if $W \in [W^*(H), \mathcal{W}(H^s))$, then $H_c(W)$ belongs to the interval (H, H^s). Taking $H = H_{t-1}$ and $W = W^*(H_{t-1})$, this implies that $H_c(W^*(H_{t-1}))$ belongs to the interval (H_{t-1}, H^s). It remains to show that the sequence converges to H^s. Since the sequence is bounded by H^s and is increasing, it converges. Let H_∞ denote this limit. We know that for all $t \geq 1$, we have that $\mathcal{P}(H_t, W^*(H_t), W^*(H_{t-1})) = C$. Given that $W^*(H)$ is continuous, it follows that $\mathcal{P}(H_\infty, W^*(H_\infty), W^*(H_\infty)) = C$. Since $\mathcal{P}(H, W^*(H), W^*(H)) > C$ for all $H < H^s$, this implies that H_∞ must equal H^s (recall that $W^*(H^s) = \mathcal{W}(H^s)$ and that $\mathcal{P}(H^s, \mathcal{W}(H^s), \mathcal{W}(H^s)) = C$).

Finally, let $W < \mathcal{W}(H^s)$ and consider the sequence $\langle W_t \rangle_{t=1}^{\infty}$ defined inductively by $W_1 = W^*(H_c(W))$ and $W_t = W^*(H_c(W_{t-1}))$. Associated with this sequence of wealth levels, is a sequence of housing levels $\langle H_t \rangle_{t=0}^{\infty}$ defined inductively by $H_t = H_c(W_{t-1})$. This sequence satisfies the equation $H_t = H_c(W^*(H_{t-1}))$ and hence converges monotonically to H^s. Given that the function $W^*(H)$ is increasing and $W^*(H^s) = \mathcal{W}(H^s)$, it follows that the sequence $\langle W_t \rangle_{t=1}^{\infty}$ converges monotonically to $\mathcal{W}(H^s)$. ■

We can use Fact A.2.4 to shed light on the nature of the function $V^*(W)$ on the interval $[W^*(H_0), \mathcal{W}(H^s))$ as defined in (34). For any $W \in [W^*(H_0), \mathcal{W}(H^s))$, we can construct a sequence $\langle W_t(W) \rangle_{t=0}^{\infty}$ that starts at W (i.e., $W_0(W) = W$) and is defined inductively by $W_t(W) = W^*(H_c(W_{t-1}(W)))$ as in Fact A.2.4. We can then write

$$V^*(W) = (1-\mu)\left[C + \frac{\underline{u}}{1-\beta}\right] + \mu\left[S\left(H_c(W_0(W))\right) + \frac{W_0(W) - \beta W_1(W)}{H_c(W_0(W))} + \beta V^*(W_1(W))\right].$$

Similarly, we can write

$$V^*(W_1(W)) = (1-\mu)\left[C + \frac{\underline{u}}{1-\beta}\right] + \mu\left[S\left(H_c(W_1(W))\right) + \frac{W_1(W) - \beta W_2(W)}{H_c(W_1(W))} + \beta V^*(W_2(W))\right].$$

Iterating, we obtain

$$V^*(W) = \sum_{t=0}^{\infty} (\mu\beta)^t \left((1-\mu) \left[C + \frac{\underline{u}}{1-\beta} \right] + \mu \left[S\left(H_c(W_t(W))\right) + \frac{W_t(W) - \beta W_{t+1}(W)}{H_c(W_t(W))} \right] \right).$$

We know from the definition of the function $H_c(W)$ that for all t

$$(1 - H_c(W_t(W)))\overline{\theta} + S\left(H_c(W_t(W))\right) + \frac{W_t(W) - \beta W_{t+1}(W)}{H_c(W_t(W))} - C(1-\beta) = \underline{u}.$$

This implies that $H_c(W_t(W)) = \mathcal{H}(\frac{W_t(W)-\beta W_{t+1}(W)}{1-\beta})$ and allows us to write

$$V^*(W) = C + \frac{\underline{u}}{1-\beta} + \sum_{t=0}^{\infty} (\mu\beta)^t \mu\overline{\theta} \left[\mathcal{H}(\frac{W_t(W) - \beta W_{t+1}(W)}{1-\beta}) - 1 \right]. \tag{A33}$$

Combining (26) and (A33), we may conclude that

$$V^*(W) = \begin{cases} C + \frac{\underline{u}}{1-\beta} + \sum_{t=0}^{\infty} (\mu\beta)^t \mu\overline{\theta} \left[\mathcal{H}(\frac{W_t(W)-\beta W_{t+1}(W)}{1-\beta}) - 1 \right] & \text{if } W \in [W^*(H_0), \mathcal{W}(H^s)) \\ C + \frac{\underline{u}}{1-\beta} + \left(\frac{\mu\overline{\theta}}{1-\mu\beta} \right) (\mathcal{H}(W) - 1) & \text{if } W \in [\mathcal{W}(H^s), \mathcal{W}(1)] \end{cases} \tag{A34}$$

There are several points to note about the $V^*(W)$ function defined in (A34). First, note that it is continuous at $W = \mathcal{W}(H^s)$, since for all t

$$\lim_{W \nearrow \mathcal{W}(H^s)} W_t(W) - \beta W_{t+1}(W) = (1-\beta)\mathcal{W}(H^s).$$

Second, the function is increasing. This is obviously true on the interval $[\mathcal{W}(H^s), \mathcal{W}(1)]$ given that $\mathcal{H}(W)$ is increasing. To see that it is true on the interval $[W^*(H_0), \mathcal{W}(H^s))$, it is easiest to write the function as

$$V^*(W) = C + \frac{\underline{u}}{1-\beta} + \sum_{t=0}^{\infty} (\mu\beta)^t \mu\overline{\theta} \left[H_c(W_t(W)) - 1 \right].$$

It follows that

$$\frac{dV^*(W)}{dW} = \sum_{t=0}^{\infty} (\mu\beta)^t \mu\overline{\theta} \frac{dH_c(W_t(W))}{dW} W_t'(W).$$

As shown in the proof of Fact A.2.4, $dH_c(W_t(W))/dW$ is positive. We claim that for all t, $W_t'(W)$ is also positive. The proof is by induction. We know that $W_0(W) = W$ and hence the result is true for $t = 0$. Now suppose the result is true for all $\tau \leq t-1$ and consider if it is true for t. By definition, we have that $W_t(W) = W^*(H_c(W_{t-1}(W)))$. Thus, we have that

$$W_t'(W) = \frac{dW^*(H_c(W_{t-1}(W))))}{dH} \frac{dH_c(W_{t-1}(W))}{dW} W_{t-1}'(W).$$

Each term on the right hand side of this expression is positive, and hence $W_t'(W)$ is positive as required.

Third, the function defined in (A34) is likely to be kinked at $W = \mathcal{W}(H^s)$. This is because

$$\frac{dV^*(W)}{dW} = \begin{cases} \mu\overline{\theta}\sum_{t=0}^{\infty}(\mu\beta)^t\,\mathcal{H}'(\frac{W_t(W)-\beta W_{t+1}(W)}{1-\beta})(\frac{W_t'(W)-\beta W_{t+1}'(W)}{1-\beta}) \text{ if } W \in [W^*(H_0), \mathcal{W}(H^s)) \\ \left(\frac{\mu\overline{\theta}}{1-\mu\beta}\right)\mathcal{H}'(W) \text{ if } W \in [\mathcal{W}(H^s), \mathcal{W}(1)] \end{cases}$$

Under the assumption of the Theorem that $V^*(W)$ is concave, it follows from the fact that

$$\mu\frac{dV^*(\mathcal{W}(H^s))}{dW} = \frac{1}{H^s}$$

that for all $W \in [W^*(H_0), \mathcal{W}(H^s))$[1]

$$\mu\frac{dV^*(W)}{dW} > \frac{1}{H^s}.$$

Finally, note that for all $W \in [W^*(H_0), \mathcal{W}(H^s))$, the second derivative of the function $V^*(W)$ is

$$\frac{d^2V^*(W)}{dW^2} = \mu\overline{\theta}\sum_{t=0}^{\infty}(\mu\beta)^t\left[\begin{array}{c}\mathcal{H}''(\frac{W_t(W)-\beta W_{t+1}(W)}{1-\beta})(\frac{W_t'(W)-\beta W_{t+1}'(W)}{1-\beta})^2 \\ +\mathcal{H}'(\frac{W_t(W)-\beta W_{t+1}(W)}{1-\beta})(\frac{W_t''(W)-\beta W_{t+1}''(W)}{1-\beta})\end{array}\right].$$

The first part of the terms that is summed is clearly negative, but the second part is harder to sign. This is why it is difficult to establish analytically that the $V^*(W)$ function in this range is concave. The assumption of the Theorem that $V^*(W)$ is concave amounts to assuming that the second part, if positive, does not fully offset the first part.

The second fact concerns the function $W_n(H)$.

Fact A.2.5. *If $H < H^s$, then $W_n(H) > W^*(H)$. Moreover, $\mathcal{P}(H, W_n(H), W^*(H)) < C$.*

Proof of Fact A.2.5. We begin with the first claim. By definition (35), we know that $\mathcal{P}(H, W_n(H), W^*(H)) \leq C$. Suppose, to the contrary, that $W_n(H) \leq W^*(H)$. Then, since $W^* \in \Psi$, we have

$$\mathcal{P}(H, W_n(H), W^*(H)) \geq \mathcal{P}(H, W^*(H), W^*(H)) > C.$$

This is a contradiction.

Turning to the second claim, note first that if $W_n(H) \geq \mathcal{W}(H^s)$, we have that

$$\begin{aligned} \mathcal{P}(H, W_n(H), W^*(H)) &\leq \mathcal{P}(H, \mathcal{W}(H^s), W^*(H)) \\ &< \mathcal{P}(H, \mathcal{W}(H^s), \mathcal{W}(H^s)) = C \end{aligned}$$

[1] It can be shown analytically (see the proof of Fact A.4.1) that, if the value function is concave and kinked at $\mathcal{W}(H^s)$, then the left hand derivative of the value function at $\mathcal{W}(H^s)$ must be equal to $\mu(1-\mu\beta)/[(1-\beta)H^s]$. Accordingly, a necessary condition for concavity is that β exceed $1/(1+\mu)$. For realistic parameterizations, this condition will be always be satisfied.

and so the result is true. Thus, we can assume that $W_n(H) < \mathcal{W}(H^s)$. We will show in this case that if $\mathcal{P}(H, W_n(H), W^*(H)) = C$ it must be that the payoff from the policies $(W_n(H), H)$ when the state is $(W^*(H), H)$ is strictly lower than the payoff from the equilibrium policies $(W^*(H_c(W^*(H))), H_c(W^*(H)))$. But this is inconsistent with equality (36) being satisfied at H, which is a contradiction. This will imply that it must be the case that $\mathcal{P}(H, W_n(H), W^*(H)) < C$.

Consider first the payoff from the policies $(W_n(H), H)$ when the state is $(W^*(H), H)$. By the first claim, we know that $W_n(H) > W^*(H)$ so that there will be new construction next period. It then follows that next period's price will be $\mathcal{P}(H, W^*(H_c(W_n(H))), W_n(H)) = C$. Moreover, the payoff from $(W_n(H), H)$ can be written as

$$(1-\mu)\left[(1-H)\overline{\theta} + \beta C + \underline{u}\frac{\beta}{1-\beta}\right] + S(H) + \frac{W^*(H) - \beta W_n(H)}{H} + \mu\beta V^*(W_n(H)). \tag{A35}$$

Since $\mathcal{P}(H, W_n(H), W^*(H))$ is equal to C, we have that

$$S(H) + \frac{W^*(H) - \beta W_n(H)}{H} = C(1-\beta) + \underline{u} - (1-H)\overline{\theta}.$$

We can therefore write (A35) as

$$C(1-\mu\beta) + \left(\frac{1-\mu\beta}{1-\beta}\right)\underline{u} + \mu\overline{\theta}(H-1) + \mu\beta V^*(W_n(H)).$$

Now note that $H = \mathcal{H}(\frac{W^*(H) - \beta W_n(H)}{1-\beta})$, so we can write this as

$$C(1-\mu\beta) + \left(\frac{1-\mu\beta}{1-\beta}\right)\underline{u} + \mu\overline{\theta}\left[\mathcal{H}(\frac{W^*(H) - \beta W_n(H)}{1-\beta}) - 1\right] + \mu\beta V^*(W_n(H)). \tag{A36}$$

Next observe from (A34) that

$$V^*(W_n(H)) = C + \frac{\underline{u}}{1-\beta} + \sum_{t=0}^{\infty}(\mu\beta)^t \mu\overline{\theta}\left[\mathcal{H}(\frac{W_t(W_n(H)) - \beta W_{t+1}(W_n(H))}{1-\beta}) - 1\right].$$

Substituting this into (A36), the payoff from $(W_n(H), H)$ can be written as

$$C + \frac{\underline{u}}{1-\beta} + \mu\overline{\theta}\left[\mathcal{H}(\frac{W^*(H) - \beta W_n(H)}{1-\beta}) - 1\right] + \sum_{t=0}^{\infty}(\mu\beta)^{t+1}\mu\overline{\theta}\left[\mathcal{H}(\frac{W_t(W_n(H)) - \beta W_{t+1}(W_n(H))}{1-\beta}) - 1\right].$$

Note also that the payoff from the equilibrium policies $(W^*(H_c(W^*(H))), H_c(W^*(H)))$ in state $(W^*(H), H)$ can be written in a similar way as

$$C + \frac{\underline{u}}{1-\beta} + \mu\overline{\theta}\left[\mathcal{H}(\frac{W^*(H) - \beta W^*(H_c(W^*(H)))}{1-\beta}) - 1\right]$$
$$+ \sum_{t=0}^{\infty}(\mu\beta)^{t+1}\mu\overline{\theta}\left[\mathcal{H}(\frac{W_t(W^*(H_c(W^*(H)))) - \beta W_{t+1}(W^*(H_c(W^*(H))))}{1-\beta}) - 1\right]$$

Moreover, we know that since

$$H = \mathcal{H}(\frac{W^*(H) - \beta W_n(H)}{1-\beta}) < H_c(W^*(H)) = \mathcal{H}(\frac{W^*(H) - \beta W^*(H_c(W^*(H)))}{1-\beta}),$$

we must have that $W_n(H) > W^*(H_c(W^*(H)))$.

Now define the function $\varphi(W)$ on the interval $[W^*(H_c(W^*(H))), W_n(H)]$ as follows:

$$\varphi(W) = C + \frac{u}{1-\beta} + \mu\overline{\theta}\left[\mathcal{H}(\frac{W^*(H) - \beta W}{1-\beta}) - 1\right] + \sum_{t=0}^{\infty} (\mu\beta)^{t+1} \mu\overline{\theta}\left[\mathcal{H}(\frac{W_t(W) - \beta W_{t+1}(W)}{1-\beta}) - 1\right]$$

If we can show that this function is decreasing in W, we will have established that $(W_n(H), H)$ must yield a smaller payoff than $(W^*(H_c(W^*(H))), H_c(W^*(H)))$ in state $(W^*(H), H)$, which contradicts equality (36).

Consider then differentiating $\varphi(W)$. Ignoring multiplicative constants, the derivative is

$$-\mathcal{H}'(\frac{W^*(H) - \beta W}{1-\beta})\beta + \sum_{t=0}^{\infty} (\mu\beta)^{t+1} \mathcal{H}'(\frac{W_t(W) - \beta W_{t+1}(W)}{1-\beta})(W_t'(W) - \beta W_{t+1}'(W)).$$

Rearranging, and using the fact that $W_0'(W) = 1$, we can write this as

$$-\beta\left[\mathcal{H}'(\frac{W^*(H) - \beta W}{1-\beta}) - \mu\mathcal{H}'(\frac{W_0(W) - \beta W_1(W)}{1-\beta})\right]$$
$$-\sum_{t=1}^{\infty} (\mu\beta)^{t} \beta\left[\mathcal{H}'(\frac{W_{t-1}(W) - \beta W_t(W)}{1-\beta}) - \mu\mathcal{H}'(\frac{W_t(W) - \beta W_{t+1}(W)}{1-\beta})\right] W_t'(W).$$

We know that for all $t \geq 1$, $W_t'(W) > 0$. In addition, we know $W^*(H) - \beta W < W_0(W) - \beta W_1(W)$ and that for all $t \geq 1$, $W_{t-1}(W) - \beta W_t(W) < W_t(W) - \beta W_{t+1}(W)$. Since $\mathcal{H}$ is concave, this implies that the above expression is negative. We conclude that the derivative is decreasing in W as required. ■

We can use this result to get a sharper characterization of $W_n(H)$. Given Fact A.2.5, we know from (35) that

$$W_n(H) = \arg\max_{W'}\left\{(1-\mu)\left[\mathcal{P}(H, W', W^*(H)) + \frac{u}{1-\beta}\right] + \mu\left[S(H) + \frac{W^*(H) - \beta W'}{H} + \beta V^*(W')\right]\right\}.$$

The derivative of the objective function is

$$\beta\left(\mu\frac{dV^*(W')}{dW} - \frac{1}{H}\right).$$

We know that for any $W' > \mathcal{W}(H^s)$ we have that

$$\mu\frac{dV^*(W')}{dW} < \mu\frac{dV^*(\mathcal{W}(H^s))}{dW} = \frac{1}{H^s} < \frac{1}{H}.$$

Accordingly, $W_n(H) \leq \mathcal{W}(H^s)$. Recall, however, that the value function may be kinked at $\mathcal{W}(H^s)$. It follows that $W_n(H)$ must either satisfy the first order condition

$$\mu \frac{dV^*(W_n(H))}{dW} = \frac{1}{H} \tag{A37}$$

or equal $\mathcal{W}(H^s)$ if it is the case that

$$\lim_{W \nearrow \mathcal{W}(H^s)} \mu \frac{dV^*(W)}{dW} \geq \frac{1}{H}.$$

The residents' problem

We are now ready to show that the policy rules defined by (19), (27), (29), (31), and (32) solve the residents' problem (A30) when the continuation value is described by the appropriate value function and next period's price of housing is determined by the appropriate equilibrium price rule evaluated at next period's state variables.

Given that the residents' policy choices could in principle involve a housing level H' greater or less than H^s, it is as well to be absolutely clear at the outset on what the appropriate value function and equilibrium price rules are. Combining the information in (28) and (33), when $H < H^s$ the appropriate value function is

$$V(g,b,H) = \begin{cases} V^*(W^*(H)) + \frac{W - W^*(H)}{H} & \text{if } W \leq W^*(H) \\ V^*(W) & \text{if } W \in (W^*(H), \mathcal{W}(1)] \end{cases} \tag{A38}$$

where the function $V^*(W)$ is as described in (A34). From (25), when $H \geq H^s$ the appropriate value function is

$$V(g,b,H) = \begin{cases} V^*(\mathcal{W}(H)) + \frac{W - \mathcal{W}(H)}{H} & \text{if } W \leq \mathcal{W}(H) \\ V^*(W) & \text{if } W \in (\mathcal{W}(H), \mathcal{W}(1)] \end{cases} \tag{A39}$$

where the function $V^*(W)$ is again as described in (A34). Combining the information in (27) and (32), when $H < H^s$ the appropriate price rule is

$$P(g,b,H) = \begin{cases} \mathcal{P}(H, W_n(H), W) & \text{if } W < W^*(H) \\ C & \text{if } W \in [W^*(H), \mathcal{W}(1)] \end{cases}. \tag{A40}$$

From (24), when $H \geq H^s$ the appropriate price rule is

$$P(g,b,H) = \begin{cases} \mathcal{P}(H, \mathcal{W}(H), W) & \text{if } W \leq \mathcal{W}(H) \\ C & \text{if } W \in (\mathcal{W}(H), \mathcal{W}(1)] \end{cases}. \tag{A41}$$

Our first observation is that the residents' policy choices are such that next period's wealth is at least as big as the threshold level $W^*(H')$ if their housing choice is less than H^s and at least as big as $\mathcal{W}(H')$ if it is greater than H^s.

Fact A.2.6. *Suppose that $H < H^s$ and let (g', W', H', P) solve problem (A30) with future price and continuation value as described in (A38), (A39), (A40), and (A41). Then, $W' \geq W^*(H')$ if $H' < H^s$ and $W' \geq \mathcal{W}(H')$ if $H' \geq H^s$.*

Proof of Fact A.2.6. Suppose first, to the contrary, that $W' < W^*(H')$ and $H' < H^s$. Then, from (A40) and (A38)

$$P(g', \beta\left(W' - cg'\right), H') = \mathcal{P}(H', W_n(H'), W')$$

and

$$V(g', \beta\left(W' - cg'\right), H') = V^*(W^*(H')) + \frac{W' - W^*(H')}{H'}.$$

Thus, the current price is

$$P = (1 - H')\overline{\theta} + B(\frac{g'/(1-\delta)}{(H')^{\alpha}}) - \frac{c\left(\frac{g'}{1-\delta} - \beta g'\right)}{H'} + \frac{W - \beta W'}{H'} + \beta\mathcal{P}(H', W_n(H'), W') - \underline{u}.$$

and the payoff is

$$(1-\mu)\left[P + \frac{\underline{u}}{1-\beta}\right] + \mu\left[B(\frac{g'/(1-\delta)}{(H')^{\alpha}}) - \frac{c\left(\frac{g'}{1-\delta} - \beta g'\right)}{H'} + \frac{W - \beta W'}{H'} + \beta\left(V^*(W^*(H')) + \frac{W' - W^*(H')}{H'}\right)\right]$$

Note that neither the price nor the payoff vary with respect to W' for any $W' < W^*(H')$. However, at $W' = W^*(H')$, the price jumps up reflecting the fact that the future price jumps from $\mathcal{P}(H', W_n(H'), W^*(H'))$ (which is less than C by Fact A.2.5) to C. If it is the case, that

$$(1 - H')\overline{\theta} + B(\frac{g'/(1-\delta)}{(H')^{\alpha}}) - \frac{c\left(\frac{g'}{1-\delta} - \beta g'\right)}{H'} + \frac{W - \beta W^*(H')}{H'} + \beta C - \underline{u} \leq C$$

Then it is clear that we can replace (g', W', H', P) with a policy in which $W' = W^*(H')$ and we will have increased the value of the objective function. Since this is a contradiction, we can assume that

$$(1 - H')\overline{\theta} + B(\frac{g'/(1-\delta)}{(H')^{\alpha}}) - \frac{c\left(\frac{g'}{1-\delta} - \beta g'\right)}{H'} + \frac{W - \beta W^*(H')}{H'} + \beta C - \underline{u} > C.$$

We can also write the payoff under (g', W', H', P) as

$$(1-\mu)\left[P + \frac{\underline{u}}{1-\beta}\right] + \mu\left[B(\frac{g'/(1-\delta)}{(H')^{\alpha}}) - \frac{c\left(\frac{g'}{1-\delta} - \beta g'\right)}{H'} + \frac{W - \beta W^*(H')}{H'} + \beta V^*(W^*(H'))\right]. \tag{A42}$$

Now choose a wealth level $\widehat{W} > W^*(H')$ which keeps the current price equal to P, but makes the future price equal C. This price satisfies

$$(1-H')\overline{\theta} + B(\frac{g'/(1-\delta)}{(H')^{\alpha}}) - \frac{c\left(\frac{g'}{1-\delta} - \beta g'\right)}{H'} + \frac{W - \beta \widehat{W}}{H'} + \beta C - \underline{u} = P. \tag{A43}$$

The policies $(g', \widehat{W}, H', P)$ yield a payoff

$$(1-\mu)\left[P + \frac{\underline{u}}{1-\beta}\right] + \mu\left[B(\frac{g'/(1-\delta)}{(H')^{\alpha}}) - \frac{c\left(\frac{g'}{1-\delta} - \beta g'\right)}{H'} + \frac{W - \beta \widehat{W}}{H'} + \beta V^*(\widehat{W})\right]. \tag{A44}$$

Comparing (A42) and (A44), we see that choosing the alternative policies $(g', \widehat{W}, H', P)$ will increase the objective function if

$$V^*(\widehat{W}) - \frac{\widehat{W}}{H'} \geq V^*(W^*(H')) - \frac{W^*(H')}{H'}.$$

Given that V^* is strictly concave, this will be true if

$$\frac{dV^*(\widehat{W})}{dW} \geq \frac{1}{H'}.$$

We know from (A37) that

$$\frac{dV^*(W_n(H'))}{dW} > \frac{1}{H'},$$

thus it is sufficient to show that $\widehat{W}$ is less than $W_n(H')$.

To establish this, note first from (A43) that

$$\begin{aligned} & (1-H')\overline{\theta} + B(\frac{g'/(1-\delta)}{(H')^{\alpha}}) - \frac{c\left(\frac{g'}{1-\delta} - \beta g'\right)}{H'} + \frac{W - \beta W^*(H')}{H'} + \beta \mathcal{P}(H', W_n(H'), W^*(H')) - \underline{u} \\ = \quad & (1-H')\overline{\theta} + B(\frac{g'/(1-\delta)}{(H')^{\alpha}}) - \frac{c\left(\frac{g'}{1-\delta} - \beta g'\right)}{H'} + \frac{W - \beta \widehat{W}}{H'} + \beta C - \underline{u}. \end{aligned}$$

Cancelling terms and dividing through by β, this implies that

$$\mathcal{P}(H', W_n(H'), W^*(H')) + \frac{\widehat{W} - W^*(H')}{H'} = C.$$

Recall that, by definition,

$$\mathcal{P}(H', W_n(H'), W^*(H')) \equiv (1-H')\overline{\theta} + S(H') + \frac{W^*(H') - \beta W_n(H')}{H'} + \beta C - \underline{u}.$$

Thus, this implies that

$$(1-H')\overline{\theta} + S(H') + \frac{\widehat{W} - \beta W_n(H')}{H'} + \beta C - \underline{u} = C$$

or, equivalently, that

$$\mathcal{P}(H', W_n(H'), \widehat{W}) = C.$$

This equality implies that $\widehat{W} < W_n(H')$. Suppose, to the contrary, that $\widehat{W} \geq W_n(H')$. Then, by Fact A.2.5 and the fact that (by definition) $W^*(H') > \mathcal{W}(H')$, we have that

$$\begin{aligned} \mathcal{P}(H', W_n(H'), \widehat{W}) &\geq \mathcal{P}(H', W_n(H'), W_n(H')) \\ &> \mathcal{P}(H', W^*(H'), W^*(H')) \\ &> \mathcal{P}(H', \mathcal{W}(H'), \mathcal{W}(H')) = C, \end{aligned}$$

which is a contradiction.

Next suppose, again contrary to the fact, that $W' < \mathcal{W}(H')$ and $H' \geq H^s$. Then using the same argument as used to establish Fact A.2.1, we can show that increasing W' to $\mathcal{W}(H')$ will not violate the constraints or change the value of the objective function in problem (A30). ■

Fact A.2.6 allows us to rewrite problem (A30) as

$$\max_{(g', W', H', P)} \left\{ \begin{array}{c} (1-\mu)\left[P + \frac{\underline{u}}{1-\beta}\right] + \mu\left[B\left(\frac{g'/(1-\delta)}{(H')^\alpha}\right) - \frac{c\left(\frac{g'}{1-\delta} - \beta g'\right)}{H'} + \frac{W - \beta W'}{H'} + \beta V^*(W')\right] \\ s.t.\ P = (1-H')\overline{\theta} + B(\frac{g'/(1-\delta)}{(H')^\alpha}) - \frac{c\left(\frac{g'}{1-\delta} - \beta g'\right)}{H'} + \frac{W - \beta W'}{H'} + \beta C - \underline{u} \\ H' \geq H \\ P \leq C \ (= \text{if } H' > H) \\ W' \geq W^*(H') \text{ if } H' < H^s \ \&\ W' \geq \mathcal{W}(H') \text{ if } H' \geq H^s. \end{array} \right\} \tag{A45}$$

Our next result shows that public good provision is efficient.

Fact A.2.7. *Suppose that $H < H^s$ and let (g', W', H', P) solve problem (A45) with $V^*(W')$ as described in (A34). Then, $g' = (1-\delta)g^o(H')$.*

Proof of Fact A.2.7. The argument is identical to the proof of Fact A.2.2. ■

Fact A.2.7 allows us to write the residents' problem as

$$\max_{(W', H')} \left\{ \begin{array}{c} (1-\mu)\left[(1-H')\overline{\theta} + \beta C - \underline{u} + \frac{\underline{u}}{1-\beta}\right] + S(H') + \frac{W - \beta W'}{H'} + \mu\beta V^*(W') \\ s.t.\ H' \geq H \\ (1-H')\overline{\theta} + S(H') + \frac{W - \beta W'}{H'} - (1-\beta)C \leq \underline{u} \ (= \text{if } H' > H) \\ W' \geq W^*(H') \text{ if } H' < H^s \ \&\ W' \geq \mathcal{W}(H') \text{ if } H' \geq H^s. \end{array} \right\}. \tag{A46}$$

Our next result ties down the optimal (W', H').

Fact A.2.8. *Suppose that $H < H^s$ and let (W', H') solve problem (A46) with $V^*(W')$ as described in (A34). Then,*

$$(W', H') = \begin{cases} (W_n(H), H) & \text{if } W < W^*(H) \\ (W^*(H_c(W)), H_c(W)) & \text{if } W \in [W^*(H), \mathcal{W}(H^s)) \\ (W, \mathcal{H}(W)) & \text{if } W \in [\mathcal{W}(H^s), \mathcal{W}(1)] \end{cases} .$$

Proof of Fact A.2.8. There are two possibilities to consider: i) the price constraint holds with equality at the optimal policies, and ii) the price constraint holds with inequality at the optimal policies. We begin with the first possibility.

Possibility i). If

$$(1 - H')\overline{\theta} + S(H') + \frac{W - \beta W'}{H'} - (1 - \beta)C = \underline{u},$$

then $H' = \mathcal{H}((W - \beta W')/(1 - \beta))$. The constraint that $H' \geq H$ then implies that $W - \beta W' \geq (1 - \beta)\mathcal{W}(H)$ which means that

$$W' \leq \frac{W - (1 - \beta)\mathcal{W}(H)}{\beta}.$$

We also need to respect the constraint that $W' \geq W^*(H')$ if $H' < H^s$ and $W' \geq \mathcal{W}(H')$ if $H' \geq H^s$. In this regard, note first that if $W \geq \mathcal{W}(H^s)$ then $(W', \mathcal{H}(\frac{W-\beta W'}{1-\beta}))$ is such that $W' \geq \mathcal{W}(\mathcal{H}(\frac{W-\beta W'}{1-\beta}))$ if and only if $W' \geq W$. Second, if $W < \mathcal{W}(H^s)$ then $(W', \mathcal{H}(\frac{W-\beta W'}{1-\beta}))$ is such that $W' \geq W^*(\mathcal{H}(\frac{W-\beta W'}{1-\beta}))$ if and only if $W' \geq W^*(H_c(W))$. Finally, note that if $W \geq \mathcal{W}(H^s)$ and $W' \geq W$, then $\mathcal{H}(\frac{W-\beta W'}{1-\beta}) \geq H^s$, while if $W < \mathcal{W}(H^s)$ and $W' \geq W^*(H_c(W))$, then $\mathcal{H}(\frac{W-\beta W'}{1-\beta}) < H^s$.

From these observations, we conclude that if $W < \mathcal{W}(H^s)$ the range of feasible W' values is

$$W' \in [W^*(H_c(W)), \frac{W - (1 - \beta)\mathcal{W}(H)}{\beta}],$$

while if $W \geq \mathcal{W}(H^s)$ the range of feasible W' values is

$$W' \in [W, \frac{W - (1 - \beta)\mathcal{W}(H)}{\beta}].$$

To characterize the optimal W', we consider first the case $W < \mathcal{W}(H^s)$. For the interval $[W^*(H_c(W)), \frac{W-(1-\beta)\mathcal{W}(H)}{\beta}]$ to be non-empty, it is necessary that

$$\beta W^*(H_c(W)) + (1 - \beta)\mathcal{W}(H) \leq W.$$

If this condition is not satisfied, then there exist no values of W' such that both $\mathcal{H}((W-\beta W')/(1-\beta)) \geq H$ and $W' \geq W^*(\mathcal{H}((W-\beta W')/(1-\beta)))$. This condition is equivalent to the requirement that $W \geq \beta W^*(H) + (1-\beta)\mathcal{W}(H)$. To see this, note that

$$\mathcal{P}(H, W^*(H), H_c^{-1}(H)) = C.$$

Thus,

$$H_c^{-1}(H) - \beta W^*(H) = (1-\beta)\mathcal{W}(H),$$

which implies that

$$H_c^{-1}(H) = \beta W^*(H) + (1-\beta)\mathcal{W}(H).$$

If $W \geq H_c^{-1}(H)$, then $H_c(W) \geq H$ and

$$\mathcal{P}(H_c(W), W^*(H_c(W)), W) = C.$$

This implies that

$$W - \beta W^*(H_c(W)) = (1-\beta)\mathcal{W}(H_c(W)) \geq (1-\beta)\mathcal{W}(H)$$

If $W < H_c^{-1}(H)$, then $H_c(W) < H$ and

$$\mathcal{P}(H_c(W), W^*(H_c(W)), W) = C.$$

This implies that

$$W - \beta W^*(H_c(W)) = (1-\beta)\mathcal{W}(H_c(W)) < (1-\beta)\mathcal{W}(H).$$

Using the fact that the price constraint binds, we can write the objective function as

$$C(1-\mu\beta) + \left(\frac{1-\mu\beta}{1-\beta}\right)\underline{u} + \mu\overline{\theta}\left[\mathcal{H}(\frac{W-\beta W'}{1-\beta}) - 1\right] + \mu\beta V^*(W')$$

Note from (A34) that the form of the function $V^*(W')$ depends on whether W' is greater than or less than $\mathcal{W}(H^s)$. This requires us to distinguish two cases: a) $W \leq \beta\mathcal{W}(H^s) + (1-\beta)\mathcal{W}(H)$; and b) $W \in (\beta\mathcal{W}(H^s) + (1-\beta)\mathcal{W}(H), \mathcal{W}(H^s))$.

In case a), $\frac{W-(1-\beta)\mathcal{W}(H)}{\beta} \leq \mathcal{W}(H^s)$. Using (A34), the optimal W' must therefore solve the problem

$$\max_{W'}\left\{\begin{array}{c} \mu\overline{\theta}\left[\mathcal{H}(\frac{W-\beta W'}{1-\beta}) - 1\right] + \sum_{t=0}^{\infty}(\mu\beta)^{t+1}\mu\overline{\theta}\left[\mathcal{H}(\frac{W_t(W')-\beta W_{t+1}(W')}{1-\beta}) - 1\right] \\ s.t.\ W' \in [W^*(H_c(W)), \frac{W-(1-\beta)\mathcal{W}(H)}{\beta}] \end{array}\right\}.$$

We claim that the objective function is decreasing in W' and hence that the solution to this problem is $W^*(H_c(W))$. Ignoring multiplicative constants, the derivative of the objective function is

$$-\mathcal{H}'(\frac{W-\beta W'}{1-\beta})\beta+\sum_{t=0}^{\infty}(\mu\beta)^{t+1}\mathcal{H}'(\frac{W_t(W')-\beta W_{t+1}(W')}{1-\beta})(W_t'(W')-\beta W_{t+1}'(W')).$$

Rearranging, and using the fact that $W_0'(W')=1$, we can write this as

$$-\beta\left[\mathcal{H}'(\frac{W-\beta W'}{1-\beta})-\mu\mathcal{H}'(\frac{W_0(W')-\beta W_1(W')}{1-\beta})\right]$$
$$-\sum_{t=1}^{\infty}(\mu\beta)^t\beta\left[\mathcal{H}'(\frac{W_{t-1}(W')-\beta W_t(W')}{1-\beta})-\mu\mathcal{H}'(\frac{W_t(W')-\beta W_{t+1}(W')}{1-\beta})\right]W_t'(W').$$

We know that for all $t\geq 1$, $W_t'(W')>0$. In addition, we know $W-\beta W'<W_0(W')-\beta W_1(W')$ and that for all $t\geq 1$, $W_{t-1}(W')-\beta W_t(W')<W_t(W')-\beta W_{t+1}(W')$. Since $\mathcal{H}$ is concave, this implies that the above expression is negative.

In case b), the optimal choice of wealth maximizes the objective function:

$$\begin{cases}\mu\overline{\theta}\left[\mathcal{H}(\frac{W-\beta W'}{1-\beta})-1\right]+\sum_{t=0}^{\infty}(\mu\beta)^{t+1}\mu\overline{\theta}\left[\mathcal{H}(\frac{W_t(W')-\beta W_{t+1}(W')}{1-\beta})-1\right] & \text{if } W'\in[W^*(H_c(W)),\mathcal{W}(H^s)) \\ \mu\overline{\theta}\left[\mathcal{H}(\frac{W-\beta W'}{1-\beta})-1\right]+\left(\frac{\mu\beta}{1-\mu\beta}\right)\mu\overline{\theta}(\mathcal{H}(W')-1) & \text{if } W'\in[\mathcal{W}(H^s),\frac{W-(1-\beta)\mathcal{W}(H)}{\beta}]\end{cases}$$

As shown in case a), this objective function is decreasing on $[W^*(H_c(W)),\mathcal{W}(H^s))$, and, as shown below, it is also decreasing on $[\mathcal{W}(H^s),\frac{W-(1-\beta)\mathcal{W}(H)}{\beta}]$. Given that the objective function is continuous at $\mathcal{W}(H^s)$, the optimal choice of wealth is $W^*(H_c(W))$.

We conclude that when $W<\mathcal{W}(H^s)$, if the price constraint holds with equality at the optimal policies, then the optimal policies are $(W^*(H_c(W)),H_c(W))$. A necessary condition for this to be the solution is that $W\geq\beta W^*(H)+(1-\beta)\mathcal{W}(H)$. Note for future reference that the payoff from this candidate solution is $V^*(W)$.

We now turn to the case $W\geq\mathcal{W}(H^s)$. For the interval $[W,\frac{W-(1-\beta)\mathcal{W}(H)}{\beta}]$ to be non-empty, it is necessary that $\mathcal{W}(H)\leq W$ which is the case if $W\geq\mathcal{W}(H^s)$. Using the fact that the price constraint binds, we can write the objective function as

$$C(1-\mu\beta)+\left(\frac{1-\mu\beta}{1-\beta}\right)\underline{u}+\mu\overline{\theta}\left[\mathcal{H}(\frac{W-\beta W'}{1-\beta})-1\right]+\mu\beta V^*(W')$$

Note from (A34) that the form of the function $V^*(W')$ depends on whether W' is greater than or less than $\mathcal{W}(H^s)$. Since $W'\geq W\geq\mathcal{W}(H^s)$, the optimal W' must solve the problem

$$\max_{W'}\left\{\begin{array}{c}\mu\overline{\theta}\left[\mathcal{H}(\frac{W-\beta W'}{1-\beta})-1\right]+\left(\frac{\mu\beta}{1-\mu\beta}\right)\mu\overline{\theta}(\mathcal{H}(W')-1) \\ s.t.\ W'\in[W,\frac{W-(1-\beta)\mathcal{W}(H)}{\beta}]\end{array}\right\}.$$

Ignoring multiplicative constants, the derivative of the objective function is

$$\frac{\mu\beta}{1-\mu\beta}\mathcal{H}'(W') - \frac{\beta}{1-\beta}\mathcal{H}'(\frac{W-\beta W'}{1-\beta}).$$

The concavity of the function $\mathcal{H}(W)$, implies that this derivative is negative for all $W' \in [W, \frac{W-(1-\beta)\mathcal{W}(H)}{\beta}]$. The optimal choice of wealth is therefore W.

We conclude that when $W \geq \mathcal{W}(H^s)$, if the price constraint holds with equality at the optimal policies, then the optimal policies are $(W, \mathcal{H}(W))$. Note for future reference that the payoff from this candidate solution is $V^*(W)$.

Possibility ii). If the price constraint holds as an inequality at the optimal policies, then $H' = H$ and $W - \beta W' < (1-\beta)\mathcal{W}(H')$. This means that

$$W' > \frac{W-(1-\beta)\mathcal{W}(H)}{\beta}.$$

Since $H < H^s$, the constraint that $W' \geq W^*(H')$ if $H' < H^s$ and $W' \geq \mathcal{W}(H')$ if $H' \geq H^s$, requires that $W' \geq W^*(H)$. The optimal choice of wealth therefore solves the problem

$$\max_{W'} \left\{ \begin{array}{c} (1-\mu)\left[(1-H)\overline{\theta} + \beta C - \underline{u} + \frac{\underline{u}}{1-\beta}\right] + S(H) + \frac{W-\beta W'}{H} + \mu\beta V^*(W') \\ W' \geq W^*(H) \end{array} \right\}$$

We claim that the solution is $W_n(H)$ as defined in (35). Notice from (23) that the objective function in this problem is equal to

$$(1-\mu)\left[\mathcal{P}(H, W', W) + \frac{\underline{u}}{1-\beta}\right] + \mu\left[S(H) + \frac{W-\beta W'}{H} + \beta V^*(W')\right].$$

Given that W enters as an additive constant and can have no impact on the solution, this is equivalent to an objective function

$$(1-\mu)\left[\mathcal{P}(H, W', W^*(H)) + \frac{\underline{u}}{1-\beta}\right] + \mu\left[S(H) + \frac{W^*(H)-\beta W'}{H} + \beta V^*(W')\right].$$

From (35) and Fact A.2.5, we know that

$$W_n(H) = \arg\max_{W'}(1-\mu)\left[\mathcal{P}(H, W', W^*(H)) + \frac{\underline{u}}{1-\beta}\right] + \mu\left[S(H) + \frac{W^*(H)-\beta W'}{H} + \beta V^*(W')\right]$$

and that $W_n(H) > W^*(H)$. Thus, the solution is $W_n(H)$ as claimed.

We conclude that if the price constraint holds as an inequality at the optimal policies, then the optimal policies are $(W_n(H), H)$. A necessary condition for this to be the solution is that $W < \beta W_n(H) + (1-\beta)\mathcal{W}(H)$. If this condition is not satisfied, then the price constraint cannot hold as an inequality when the policies are $(W_n(H), H)$. The payoff from this candidate solution is

$$(1-\mu)\left[\mathcal{P}(H, W_n(H), W) + \frac{u}{1-\beta}\right] + \mu\left[S(H) + \frac{W - \beta W_n(H)}{H} + \beta V^*(W_n(H))\right]$$

Which possibility arises? Having understood the two possibilities, we can now analyze which one arises. Possibility ii) involves the policies $(W_n(H), H)$. A necessary condition for this to be the solution is that $W < \beta W_n(H) + (1-\beta)\mathcal{W}(H)$. Possibility i) involves the policies $(W^*(H_c(W)), H_c(W))$ when $W < \mathcal{W}(H^s)$ and the policies $(W, \mathcal{H}(W))$ when $W \geq \mathcal{W}(H^s)$. A necessary condition for possibility i) to be the solution when $W < \mathcal{W}(H^s)$ is that $W \geq \beta W^*(H) + (1-\beta)\mathcal{W}(H)$.

Given that $W_n(H) \leq \mathcal{W}(H^s)$, when $W \geq \mathcal{W}(H^s)$, it must be the case that $W > \beta W_n(H) + (1-\beta)\mathcal{W}(H)$ and thus the solution must be possibility i); that is,

$$(W', H') = (W, \mathcal{H}(W)) \text{ if } W \geq \mathcal{W}(H^s).$$

We may therefore focus on the case in which $W < \mathcal{W}(H^s)$. Given that by Fact A.2.5, $W_n(H) > W^*(H)$, we can conclude that the solution is $(W_n(H), H)$ if $W < \beta W^*(H) + (1-\beta)\mathcal{W}(H)$ and $(W^*(H_c(W)), H_c(W))$ if $W \geq \beta W_n(H) + (1-\beta)\mathcal{W}(H)$. For values of W in the interval $[\beta W^*(H) + (1-\beta)\mathcal{W}(H), \beta W_n(H) + (1-\beta)\mathcal{W}(H))$ both possibilities are feasible. Thus, which possibility is optimal depends on a comparison of the payoffs. We claim that:

$$(W', H') = \begin{cases} (W_n(H), H) & \text{if } W < W^*(H) \\ (W^*(H_c(W)), H_c(W)) & \text{if } W \in [W^*(H), \mathcal{W}(H^s)) \end{cases}.$$

Define the function $\varphi(W; H)$ on the interval $[\beta W^*(H) + (1-\beta)\mathcal{W}(H), \beta W_n(H) + (1-\beta)\mathcal{W}(H))$ to be equal to the difference between the payoffs from the two candidate solutions; that is,

$$\varphi(W; H) = V^*(W) - (1-\mu)\left[\mathcal{P}(H, W_n(H), W) + \frac{u}{1-\beta}\right] - \mu\left[S(H) + \frac{W - \beta W_n(H)}{H} + \beta V^*(W_n(H))\right].$$

Condition (36) implies that $\varphi(W^*(H); H) = 0$. Thus, it is sufficient to show that $\varphi(W; H)$ is increasing in W. Differentiating, we have that

$$\frac{d\varphi(W; H)}{dW} = \frac{dV^*(W)}{dW} - \frac{1}{H}.$$

We know that $W < W_n(H)$ and from (A37) that

$$\frac{dV^*(W_n(H))}{dW} > \frac{1}{H}.$$

Thus, it follows from the concavity of $V^*(W)$, that $\varphi(W;H)$ is increasing in W as required. ■

Using Facts A.2.7 and A.2.8, it is now clear that the residents want to follow the equilibrium policy rules described in (19), (27), (29), (31), and (32).

Verifying form of value function

It remains to verify that the value function as described by (33) and (A34) satisfies (9). Suppose first that (g,b,H) is such that $W \geq W^*(H)$. Then, we have that

$$\begin{aligned}
& (1-\mu)\left[P(\cdot)+\frac{u}{1-\beta}\right]+\mu\left[B\left(\frac{g'(\cdot)/(1-\delta)}{H'(\cdot)^{\alpha}}\right)-T(\cdot)+\beta V(g'(\cdot),b'(\cdot),H'(\cdot))\right] \\
= \; & (1-\mu)\left[C+\frac{u}{1-\beta}\right]+\mu\left[S(H_c(W))+\frac{W-\beta W^*(H_c(W))}{H_c(W)}+\beta V^*(W^*(H_c(W)))\right] \\
= \; & V^*(W),
\end{aligned}$$

as required. Now suppose that (g,b,H) is such that $W < W^*(H)$. Then, using (36), we have that

$$\begin{aligned}
& (1-\mu)\left[P(\cdot)+\frac{u}{1-\beta}\right]+\mu\left[B\left(\frac{g'(\cdot)/(1-\delta)}{H'(\cdot)^{\alpha}}\right)-T(\cdot)+\beta V(g'(\cdot),b'(\cdot),H'(\cdot))\right] \\
= \; & (1-\mu)\left[\mathcal{P}(H,W_n(H),W)+\frac{u}{1-\beta}\right]+\mu\left[S(H)+\frac{W-\beta W_n(H)}{H}+\beta V^*(W_n(H))\right] \\
= \; & (1-\mu)\left[\mathcal{P}(H,W_n(H),W^*(H))+\frac{u}{1-\beta}\right]+\mu\left[S(H)+\frac{W^*(H)-\beta W_n(H)}{H}+\beta V^*(W_n(H))\right] \\
& +\frac{W-W^*(H)}{H} \\
= \; & V^*(W^*(H))+\frac{W-W^*(H)}{H},
\end{aligned}$$

as required. ■

Appendix 3: Proofs of Propositions 5, 6, and 7

Proof of Proposition 5

We begin by verifying the claims about housing, public good investment, and wealth. Given that $H_0 < H^s$ and that W_0 exceeds $\mathcal{W}(H^s)$, it follows from (19) and (27) that $(g_1, H_1) = ((1-\delta)g^o(\mathcal{H}(W_0)), \mathcal{H}(W_0))$ and

$$W_1 = cg_1 - (1+\rho)b_1 = cg_1 - (1+\rho)\left(\frac{c(1-\delta)g^o(\mathcal{H}(W_0)) - W_0}{1+\rho}\right) = W_0.$$

This means that $(W_1, H_1) = (W_0, \mathcal{H}(W_0))$. It follows that $W_1 = \mathcal{W}(H_1)$ and, since W_0 exceeds $\mathcal{W}(H^s)$, that $H_1 \geq H^s$. Accordingly, it follows from (19) and (21) that $(g_2, H_2) = ((1-\delta)g^o(\mathcal{H}(W_1)), \mathcal{H}(W_1))$. In addition, from (22), $W_2 = W_1$. Repeated application of this argument implies that for all $t \geq 1$, $(W_t, H_t, g_t) = (W_1, \mathcal{H}(W_1), (1-\delta)g^o(\mathcal{H}(W_1))) = (W_0, \mathcal{H}(W_0), (1-\delta)g^o(\mathcal{H}(W_0)))$.

Turning to what is happening to debt, from (27), we have that

$$\begin{aligned} b_1 - b_0 &= \frac{c(1-\delta)g^o(\mathcal{H}(W_0)) - W_0}{1+\rho} - \frac{cg_0 - W_0}{1+\rho} \\ &= \frac{c(1-\delta)g^o(\mathcal{H}(W_0)) - cg_0}{1+\rho}. \end{aligned}$$

Furthermore, from (22), for all $t \geq 2$

$$b_{t+1} - b_t = \frac{c(1-\delta)g^o(\mathcal{H}(W_0)) - W_0}{1+\rho} - \frac{c(1-\delta)g^o(\mathcal{H}(W_0)) - W_0}{1+\rho} = 0.$$

Thus, the value of outstanding debt increases by $c\left((1-\delta)g^o(\mathcal{H}(W_0)) - g_0\right)$ in period 0 and then remains constant. Finally, given that W_0 exceeds $\mathcal{W}(H^s)$ and that for all periods $t \geq 1$, $W_t = \mathcal{W}(H_t)$ and $H_t \geq H^s$, (27) and (32) imply that the price of housing is constant at the construction cost C. ■

Proof of Proposition 6

Given that $H_0 < H^s$ and $W_0 \in [W^*(H_0), \mathcal{W}(H^s))$, (29) and (31) tell us that $H_1 = H_c(W_0)$ and $W_1 = W^*(H_1)$. By Fact A.2.4, we have that $H_1 < H^s$ and $W_1 \in [W^*(H_0), \mathcal{W}(H^s))$ and so, by (29) and (31), $H_2 = H_c(W_1)$ and $W_2 = W^*(H_2)$. Repeated application of this logic reveals that in each period t beyond period 1, H_t is equal to $H_c(W^*(H_{t-1}))$ and W_t is equal to $W^*(H_t)$. By Fact A.2.4, the sequence of housing levels $\langle H_t \rangle_{t=0}^{\infty}$ is increasing and converges asymptotically to H^s. This implies that new construction takes place in each period and that the size of the community approaches H^s. Given that the threshold wealth function $W^*(\cdot)$ is increasing and that

$W^*(H^s) = \mathcal{W}(H^s)$, this implies that wealth is increasing and converging asymptotically to $\mathcal{W}(H^s)$. Equation (19) tells us that in each period $t \geq 1$, $g_t = (1-\delta)g^o(H_{t+1})$ implying that in each period t the public good level enjoyed by the residents is $g^o(H_{t+1})$.

Regarding the value of outstanding debt, equation (31) implies that for all t

$$b_{t+1} - b_t = \frac{cg_{t+1} - W_{t+1}}{1+\rho} - \frac{cg_t - W_t}{1+\rho}.$$

Thus, since $W_{t+1} > W_t$, we have that

$$\begin{aligned}(1+\rho)\left(b_{t+1} - b_t\right) &= c(g_{t+1} - g_t) - (W_{t+1} - W_t) \\ &< c(g_{t+1} - g_t),\end{aligned}$$

as required. Equation (24) implies that the price of housing is equal to the construction cost in each period. ■

Proof of Proposition 7

Given that $H_0 < H^s$ and $W_0 < W^*(H_0)$, (29) and (31) tell us that $H_1 = H_0$ and $W_1 = W_n(H_0)$. As shown in Fact A.2.5, the fact that $H_0 < H^s$, implies that $W_n(H_0) > W^*(H_0)$ and that $P(H_0, W_n(H_0), W_0) < C$. We also know from the discussion following Fact A.2.5, that $W_n(H_0) \leq \mathcal{W}(H^s)$. If $W_n(H_0) < \mathcal{W}(H^s)$, we can set $W_0 = W_n(H_0)$ and reapply the arguments made to establish Proposition 6. If $W_n(H_0) = \mathcal{W}(H^s)$, then we can reapply the arguments made in the proof of Proposition 5. ■

Appendix 4: Numerical Analysis

In this Appendix we investigate the existence of equilibrium in our model using numerical methods. We first describe our parametrization strategy, and in particular, the set of parameters over which our investigation is conducted. We then describe our method for checking Assumptions 1-2, constructing equilibrium objects, and analyzing whether these objects have the required properties.

Parameters

To start we need to specify the public good benefit function. We assume that it is a power function: $B(x) = B_0 x^{\sigma}/\sigma$, where σ is positive but less than one. With this assumption, there are ten parameters in our model:

β	δ	c	$\underline{u}$	$\bar{\theta}$	μ	σ	α	B_0	C
$\frac{1}{1.06}$	0.1	1	0	1	$(0,1)$	$(0,1)$	$[0,1]$	?	?

Parameters β and δ are commonly used in the literature and we set them accordingly. In particular, we set the discount rate β equal to $1/1.06$ and the depreciation rate of the public good δ equal to 0.1.

Parameters c, $\underline{u}$, and $\bar{\theta}$ can be normalized. We set them equal to 1, 0, and 1, respectively. To see that the cost of the public good c can be normalized note that the optimized public good surplus function defined in (12) can be written as $S(H) = s_0 \cdot H^{s_1}$ where $s_0 = \frac{1-\sigma}{\sigma} B_0 \left[\frac{B_0}{c[1-\beta(1-\delta)]}\right]^{\frac{\sigma}{1-\sigma}}$ and $s_1 = (1-\alpha)\frac{\sigma}{1-\sigma}$. Thus, while c influences the level of surplus obtained from any given housing stock, any change in c can be mimicked by an appropriate change in the benefit function parameter B_0. The utility households obtain from living outside the community $\underline{u}$ can be normalized, because it does not play an independent role. It is both intuitive and straightforward to verify that what matters is $\underline{u} + (1-\beta)C$. Thus, any change in $\underline{u}$ can be mimicked by an appropriate change in the housing price C. Finally, to see that the upper bound on the preference distribution $\bar{\theta}$ can be normalized, take a given parameterization incorporating the normalizations to parameters c and $\underline{u}$. Consider now a different parameterization that keeps all parameters unchanged except $\bar{\theta}$ is set to one, B_0 is adjusted so that s_0 is divided by $\bar{\theta}$, and C is divided by $\bar{\theta}$. Such a parameterization will generate a model that is isomorphic to the original model: all value functions, prices and wealth levels will equal their respective original values divided by $\bar{\theta}$.

This leaves five parameters: μ, σ, α, B_0, and C. The first three range between 0 and 1 and have simple interpretations: μ reflects population turnover; σ the concavity of public good benefits;

and α the congestibility of the public good. Accordingly, it is easy to specify a priori a range of plausible values for these parameters. This is not the case for the remaining two parameters B_0 and C. However, it is clear that, given the rest of the parameters, only some values of B_0 and C will yield interesting models. As the construction cost, C, increases, building new houses in the community becomes less desirable and, if the cost is too high, even a utilitarian planner would choose not to build. At the same time, increasing B_0 makes the community more desirable, so that, for high enough values of B_0, the planner's solution would be simply to build sufficient houses to accommodate all potential residents.

The way we deal with the pair (B_0, C) is to note that, given the rest of the parameters, they determine both the value of H^o and the value of H^s. Recall that H^o is the socially optimal housing level and H^s is the steady state level when there is public wealth accumulation. We are only interested in parameterizations in which both these housing levels lie between 0 and 1 and H^s is less than H^o. Accordingly, our approach is to specify H^s and H^o directly along with the other parameter values and then check to see whether there are underlying values B_0 and C which generate these housing levels.

In sum, we have five fixed parameters $(\beta, \delta, c, \underline{u}, \bar{\theta})$ and five parameters which vary $(\mu, \sigma, \alpha, B_0, C)$. For the variable parameters, we let μ take ten values: from 0.90 to 0.99 with an increment of 0.01; σ take nine values: from 0.1 to 0.9 with an increment of 0.1; and α take ten values from 0 to 0.9 with an increment of 0.1. Finally, for a given set of values for μ, σ, and α, we entertain eighty one possible pairs of (B_0, C) that correspond to eighty one pairs of (H^s, H^o). To get these eighty one pairs, we allow H^o to vary from 0.1 to 0.9 with an increment of 0.1 and, for each such H^o, we consider nine values of H^s of the form $H^s = rH^o$, where r varies from 0.1 to 0.9.[2] Based on this construct, we find 20878 different parameterizations that yield positive (B_0, C) and, for which there exists an interval of initial housing levels $[\underline{H}, H^s)$, that satisfy Assumptions 1-2. These are the relevant parameterizations for our purposes.

For each of these parameterizations, we implement the numerical procedure described below. We find that:

1. In 95.1% (19854) of the cases there exists an equilibrium threshold wealth function for all initial housing levels H_0 in the interval $[\underline{H}, H^s)$.

[2] The underlying logic here is that, given our other parameter values, the expressions for H^o and H^s provide two equations in two unknowns, (B_0, C), that we can solve for. Only in one instance - when $\alpha = 1$ and, hence, the optimized surplus function is constant - this logic breaks down because the two equations become colinear in B_0 and C. In this case, which we treat separately, we simply set B_0 to zero and set C to match the candidate value of H^o. We then check whether the resulting value of H^s is admissible.

2. In 4.9% (1024) of the cases there exists $\hat{H} \in (\underline{H}, H^s)$ such that there exists an equilibrium threshold wealth function for all initial housing levels H_0 in the interval $[\hat{H}, H^s)$.

The key point to note is that for 95.1% of these, there exists an equilibrium threshold wealth function for *any* of the initial housing levels H_0 that satisfy Assumptions 1-2.[3] Thus, for the overwhelming majority of the relevant parameterizations, existence of an equilibrium threshold wealth function is not problematic. In the remaining 1024 cases, there exists an equilibrium threshold wealth function for only *some* of the initial housing levels H_0 that satisfy Assumptions 1-2. For a range of initial housing levels, an equilibrium threshold wealth function does not exist.

Most of these problematic cases feature high μ. In fact, 880 of the 1024 cases have $\mu = 0.99$.[4] We have investigated what is going on in a handful of randomly picked cases in this category. In all these cases, there does exist an equilibrium. Moreover, it involves a threshold wealth function W^* and is described by $\mathcal{E}(W^*)$. However, the threshold wealth function does not belong to the set Ψ and, moreover, the associated function $V^*(W)$ is not concave.[5]

Figure 4 illustrates one of these problematic cases. The associated values of the parameters (μ, σ, α) are $(0.99, 0.5, 1)$. The values of (B_0, C) are chosen to generate a (H^s, H^o) pair equal to $(0.05, 0.1)$. Panel A illustrates the threshold wealth function $W^*(H)$ along with the function $\mathcal{W}(H)$. Notice that $W^*(H)$ is kinked at housing level 0.037 and is not differentiable. Moreover, as illustrated, at housing level 0.037, $W^*(H)$ is equal to $\mathcal{W}(H)$, as opposed to exceeding it. Panel B illustrates the associated $V^*(W)$ function. Between wealth levels 0.0036 and 0.008, the function is not concave. Panel C illustrates the function $W_n(H)$ along with the function $W^*(H)$. Notice that at housing level 0.037, $W_n(H)$ is equal to $W^*(H)$ and hence equal to $\mathcal{W}(H)$.

What is happening in this equilibrium is that at housing level 0.037, if the community is endowed with wealth $\mathcal{W}(H)$, the residents are not willing to build wealth to attract new residents. This is the case despite the fact that the housing level is less than H^s. In the case with commitment, H^s is the smallest housing level such that, if the community is endowed with wealth $\mathcal{W}(H)$, current residents would never choose to accumulate more wealth to attract new residents. However, in the commitment case, increasing wealth by ϵ will simply increase the future housing level to $\mathcal{H}(\mathcal{W}(H)+$

[3] When $\alpha = 1$, this is true in 93.33% of the cases.

[4] Another noteworthy pattern is that the problematic cases happen mostly for values of r (the ratio of H^s to H^o) in the medium-high range. There are 444 cases with $r = 0.7$; 182 cases with $r = 0.6$; and 105 cases with $r = 0.5$. However, there are only 48 cases for $r = 0.8$. Exactly why r matters is unclear.

[5] Because the concavity of $V^*(W)$ is lost, we cannot use the Theorem to conclude that the candidate equilibrium $\mathcal{E}(W^*)$ associated with W^* is indeed an equilibrium. Instead, we rely on numerical methods to confirm that $\mathcal{E}(W^*)$ is indeed an equilbrium.

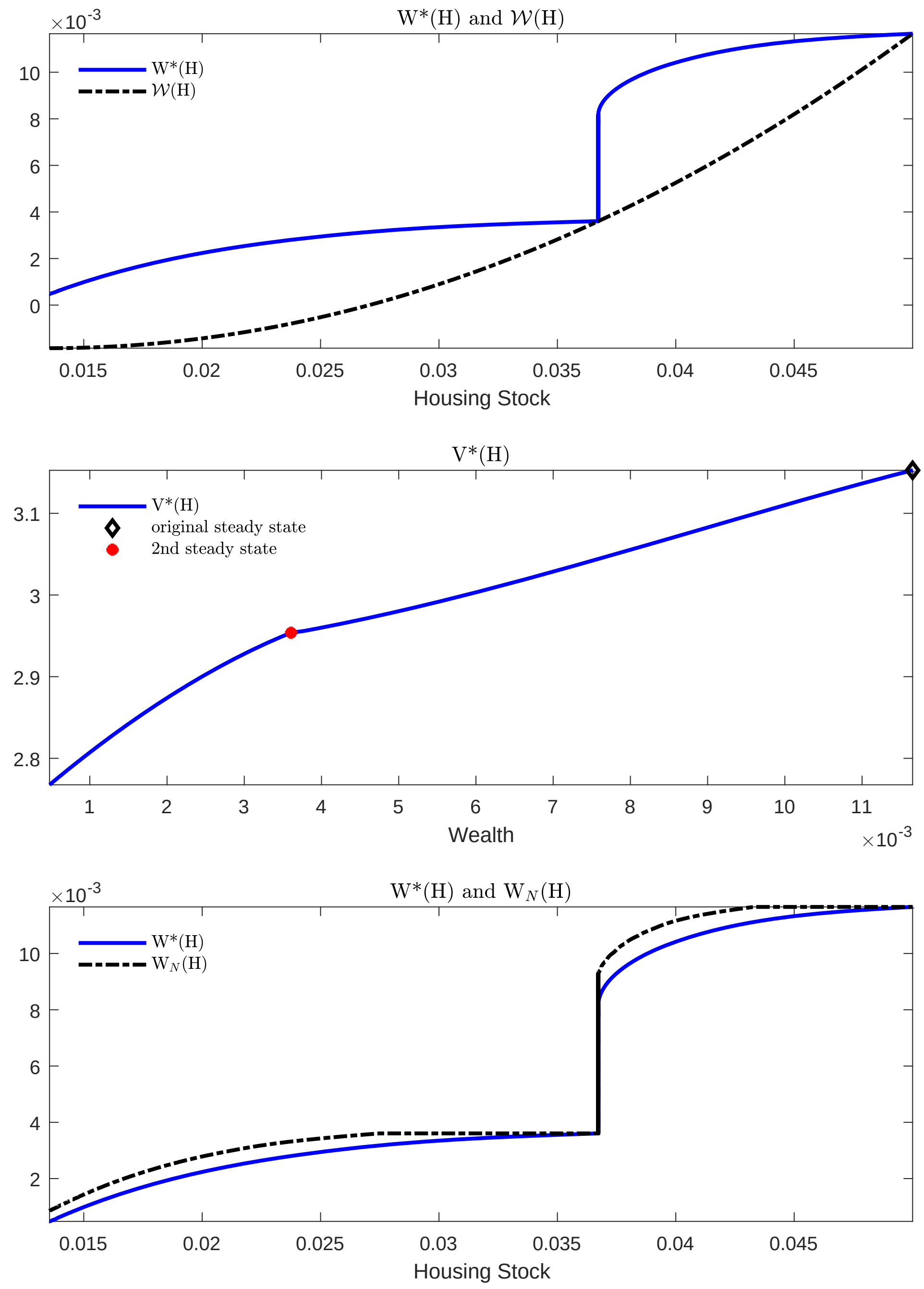

Figure 4: A problematic case

ϵ) and future wealth to $\mathcal{W}(H) + \epsilon$. At housing level 0.037, the future consequences of increasing wealth marginally are much more complicated and determined by the behavior of the threshold wealth function $W^*(H)$ between 0.037 and H^s. Evidently, these consequences are less advantageous to current residents than simply increasing the future housing level to $\mathcal{H}(\mathcal{W}(H) + \epsilon)$ and future wealth to $\mathcal{W}(H) + \epsilon$. The value of these changes is reflected in the derivative of the value function at wealth level $\mathcal{W}(0.037)$. At the kink, it must be the case that $\lim_{W \searrow \mathcal{W}(0.037)} \mu dV^*(W)/dW$ is less than 1/0.037.

Substantively, the evolution of the community in this equilibrium differs from that described in Propositions 8 and 9. If the community's initial wealth W_0 is less than $\mathcal{W}(0.037)$ it will converge asymptotically to $(\mathcal{W}(0.037), 0.037)$ as opposed to $(\mathcal{W}(H^s), H^s)$. Qualitatively, the way in which the community develops remains the same, it is just that it approaches a different point. This point involves less development, so the extent to which the community will be undersized is enhanced. It should be noted, however, that the steady state $(\mathcal{W}(0.037), 0.037)$ is not stable: any positive wealth shock will result in the community converging to $(\mathcal{W}(H^s), H^s)$.

Numerical Procedure

For a given set of parameters values we take a 100000 point uniform grid over [0, 1]. We find the smallest point $\underline{H}$ at which Assumptions 1 and 2 are satisfied. If such a point does not exist, we stop. Otherwise, we search for the smallest H_0 greater or equal to $\underline{H}$ for which an equilibrium threshold wealth function can be shown to exist. If this smallest H_0 is equal to $\underline{H}$ then, for these parameter values, an equilibrium threshold wealth function exists whenever the Assumptions are satisfied.

Constructing equilibrium

Showing that an equilibrium threshold wealth function exists for a given set of parameter values and initial conditions (H_0, W_0) amounts to showing that we can construct an increasing function $W^*(H)$ on $[H_0, H^s]$ such that: i) $W^*(H) > \mathcal{W}(H)$ for all $H \in [H_0, H^s)$; ii) $W^*(H^s) = \mathcal{W}(H^s)$; iii) the resulting $V^*(W)$ is increasing and concave; and iv) the indifference condition is satisfied. Two notes are in order. Rather than constructing function $W^*(H)$ directly it is more convenient to obtain it indirectly by constructing its inverse function, $H^*(W)$. This naturally implies that lower bound on W's, $\underline{W}$, for which the function $H^*(W)$ should be constructed must be such that $H^*(\underline{W}) \leq H_0$.

We assume and ex-post confirm that $V^*(W)$ is kinked at $\mathcal{W}(H^s)$. This implies, as shown in

Fact A.4.1 below, that $W_n(H) = \mathcal{W}(H^s)$ on $[H^s \frac{(1-\beta)}{\mu(1-\mu\beta)}, H^s)$. We use this fact in the first leg of our procedure.

Leg 1. We start by constructing the equilibrium objects in the neighborhood of the steady state $(\mathcal{W}(H^s), H^s)$. We rely on knowledge of $\lim_{W \nearrow \mathcal{W}(H^s)} \frac{dV^*(W)}{dW}$ (established in the proof of Fact A.4.1 below) to construct an approximation to the function $V^*(W)$ in this neighborhood. In particular:

1. For H close to H^s, $W = W^*(H)$ should be close to $\mathcal{W}(H^s)$. Consider interval $[W_I, \mathcal{W}(H^s)]$, where $W_I = \mathcal{W}(H^s)(1-\varepsilon)$ and ε is set to be small: specifically, $\varepsilon = 5 \cdot 10^{-4}$.

2. Take a twenty one point uniform grid on $[W_I, \mathcal{W}(H^s)]$. Call this grid $\vec{W}_I$. For each point on the grid, approximate function $V^*(W)$ using its first order Taylor expansion at $\mathcal{W}(H^s)$:

$$V^*(W) = V^*(\mathcal{W}(H^s)) + \lim_{W \nearrow \mathcal{W}(H^s)} \frac{dV^*(W)}{dW}(W - \mathcal{W}(H^s)),$$

where $\lim_{W \nearrow \mathcal{W}(H^s)} \frac{dV^*(W)}{dW}$ is as computed in the proof of Fact A.4.1 below.

3. For each $W \in \vec{W}_I$ construct $H^*(W)$ as the solution to the following indifference condition:

$$V^*(W) = \mathcal{P}(H^*(W), \mathcal{W}(H^s), W) + \mu\bar{\theta}(H^*(W) - 1) + \mu\beta(V^*(\mathcal{W}(H^s)) - C).$$

If for any W, $H^*(W)$ is lower than $H^s \frac{(1-\beta)}{\mu(1-\mu\beta)}$, reset ε to a smaller number, e.g. $\varepsilon = \varepsilon/2$, and repeat Steps 2-3.

4.

- Verify that $H^*(W)$ is increasing, that $H^*(W) > \mathcal{H}(W)$, and that $V^*(W)$ is increasing and concave on $[W_I, \mathcal{W}(H^s)]$. To do this, check that for any point A on $\vec{W}_I$

  $$(1+\epsilon)H^*(A) > \mathcal{H}(A);$$

  check that for any two adjacent points on $\vec{W}_I$, $A < B$,

  $$(1+\epsilon)H^*(B) \geq H^*(A) \text{ and } (1+\epsilon)V^*(B) \geq V^*(A);$$

  and check that for any three adjacent points on $\vec{W}_I$, $A < B < C$,

  $$(1+\epsilon)V^*(B) \geq \alpha V^*(A) + (1-\alpha)V^*(C),$$

  where $\alpha = \frac{B-A}{C-A}$ and $\epsilon = 10^{-4}$. A small ϵ is necessary to avoid interpreting numerical error in evaluating V when (a) it is near linear and (b) grid points are too close to each other, as a failure of concavity. We now have constructed $H^*(W)$ on the interval $[W_I, \mathcal{W}(H^s)]$. Observe that, by construction, we have that $H^*(\mathcal{W}(H^s)) = H^s$ and that the indifference condition (36) is satisfied for all W's on the grid.

- If not, find $\underline{W} \in [W_I, W^s]$ such that the four conditions above are satisfied on $[\underline{W}, \mathcal{W}(H^s)]$ and stop. Equilibrium exists for all $H_0 \geq \max(H^*(\underline{W}), \underline{H})$.

It is important to note here that we have built the equilibrium objects only on our grid. In what follows, when necessary, to construct either H^* or V^* at any other point on $[W_I, W(H^s)]$ we will use a shape preserving (cubic Hermite) interpolant of the relevant function.

Leg 2. We now extend our construct to the left using the knowledge of $H^*(W)$ and $V^*(W)$ on $\vec{W}_I$. The idea is to build $H^*(W)$ and $V^*(W)$ at wealth levels for which we know $H_c(W)$ and $W^*\left(H_c(W)\right)$. Clearly, the lowest value of wealth for which this can be done is the wealth level for which $H_c(W) = H^*(W_I)$. Call that point W_{II}. Consider now interval $[W_{II}, W_I]$. By construction, we have that

$$\mathcal{P}(H^*(W_I), W_I, W_{II}) = C.$$

1. For each $W_1 \in \vec{W}_I \backslash \mathcal{W}(H^s)$ find $W_2 = W_2(W_1)$ (this is an abuse of notation, but we do so to avoid adding more notation) such that

$$\mathcal{P}(H^*(W_1), W_1, W_2) = C.$$

Note that $W_{II} = W_2(W_I)$. Call this collection of points $\vec{W}_{II}$. If all elements of $\vec{W}_{II} < W_I$ proceed to the next step. Otherwise, return to Leg 1, set $\varepsilon = \varepsilon/2$ and proceed.

2. For each $W_2 \in \vec{W}_{II}$ construct function $V^*(W)$ as follows:

$$V^*(W_2) = C + \mu\theta(H^*(W_1) - 1) + \mu\beta(V^*(W_1) - C).$$

where W_1 is such that $W_2 = W_2(W_1)$.

3. For each $W_2 \in \vec{W}_{II}$ construct function $H^*(W)$ as the solution to the following indifference condition:

$$V^*(W_2) = \max_{W_n > W_1} \left\{ \begin{array}{c} \mathcal{P}(H, W_n, W_2)) + \mu\overline{\theta}(H-1) + \mu\beta(\tilde{V}(W_n) - C) \\ s.t.\ \mathcal{P}(H, W_n, W_2) \leq C \end{array} \right\}$$

where W_1 is such that $W_2 = W_2(W_1)$ and $\tilde{V}(W_n)$ is a shape preserving (cubic Hermite) interpolant of $V^*(W)$ on $[W_I, \mathcal{W}(H^s)]$.[6]

[6] Again, we use interpolant here because we have only constructed V^* at the grid points, but the search for W_n is over the entire interval $(W_1, \mathcal{W}(H^s)]$.

The indifference condition above differs from the one in (36) because of the constraint that $W_n > W_1$. This is without loss of generality because, as shown in Fact A.4.3 below, all wealth levels W_n, for which the inequality $\mathcal{P}(H, W_n, W_2) \leq C$ holds, will necessarily be larger than W_1. Indeed, Fact A.4.3 states that $W^*(H_c(W^*(H))) < W_n(H)$, and hence that $W_1 = W^*(H_c(W_2))$ should be less than $W_n(H^*(W_2))$.

4.

- Verify that $H^*(W)$ is increasing, that $H^*(W) > \mathcal{H}(W)$, and that $V(W)$ is increasing and concave on $[W_{II}, \mathcal{W}(H^s)]$. To do this, check that for any point A on $\vec{W}_{II} \cup \vec{W}_I$

  $$(1+\epsilon)H^*(A) > \mathcal{H}(A);$$

  check that for any two adjacent points on $\vec{W}_{II} \cup \vec{W}_I$, $A < B$,

  $$(1+\epsilon)H^*(B) \geq H^*(A) \text{ and } (1+\epsilon)V^*(B) \geq V^*(A);$$

  and check that for any three adjacent points on $\vec{W}_{II} \cup \vec{W}_I$, $A < B < C$,

  $$(1+\epsilon)V^*(B) \geq \alpha V^*(A) + (1-\alpha)V^*(C),$$

  where $\alpha = \frac{B-A}{C-A}$ and $\epsilon = 10^{-4}$.

  Note that by construction we have that $H^*(\mathcal{W}(H^s)) = H^s$ and that the indifference condition (36) is satisfied for all W's on the grid.

- If not, find $\underline{W} \in [W_{II}, W_I]$ such that the four conditions above are satisfied on $[\underline{W}, W^s]$ and stop. Equilibrium exists for all $H_0 \geq \max(H^*(\underline{W}), \underline{H})$.

Leg 3. We now extend our construct to the left using the knowledge of $H^*(W)$ and $V^*(W)$ on $\vec{W}_{II} \cup \vec{W}_I$. The idea again is to build $H^*(W)$ and $V^*(W)$ at wealth levels for which we know $H_c(W)$ and $W_c(W)$. Clearly, the lowest value of wealth for which this can be done is the wealth level for which $H_c(W) = H^*(W_{II})$. Call that point W_{III}. By construction, we have that

$$\mathcal{P}(H^*(W_{II}), W_{II}, W_{III}) = C.$$

1. For each $W_2 \in \vec{W}_{II}$ find $W_3(W_2)$ such that

$$\mathcal{P}(H^*(W_2), W_2, W_3) = C.$$

Note that $W_{III} = W_3(W_{II})$. Call this collection of points $\vec{W}_{III}$.

2. For each $W_3 \in \vec{W}_{III}$ construct function $V^*(W)$ as follows:

$$V^*(W_3) = C + \mu\theta(H^*(W_2) - 1) + \mu\beta(V^*(W_2) - C).$$

where W_2 is such that $W_3 = W_3(W_2)$.

3. For each $W_3 \in \vec{W}_{III}$ construct function $H^*(W)$ as the solution to the following indifference condition:

$$V^*(W_3) = \max_{W_n > W_2} \left\{ \begin{array}{c} \mathcal{P}(H, W_n, W_3)) + \mu\overline{\theta}(H-1) + \mu\beta(\tilde{V}(W_n) - C) \\ s.t.\ \mathcal{P}(H, W_n, W_3) \leq C \end{array} \right\}$$

where W_2 is such that $W_3 = W_3(W_2)$ and $\tilde{V}(W_n)$ is a shape preserving (cubic Hermite) interpolant of $V^*(W)$ on $[W_{II}, \mathcal{W}(H^s)]$.

Note here that we do not require that the wealth levels in $\vec{W}_{III}$ are less than W_{II}. Even if an element of $\vec{W}_{III}$ is larger than W_{II}, we keep it, as it provides an additional point where we constructed the values of V^* and H^*. This implies there is an asymmetry between Leg 2 and Leg 3 because in Leg 2 we required all wealth levels in $\vec{W}_{II}$ to be less than W_I. The reason for this asymmetry is to ensure that the initial range $[W_I, \mathcal{W}(H^s)]$ where we use the Taylor approximation to construct $V^*$and hence, H^*, is small relative to the "length of the step" between the values of W in successive Legs.

4.

- Verify that $H^*(W)$ is increasing, that $H^*(W) > \mathcal{H}(W)$, and that $V(W)$ is increasing and concave on $[W_{III}, \mathcal{W}(H^s)]$. To do this, check that for any point A on $\vec{W}_{III} \cup \vec{W}_{II} \cup \vec{W}_I$

$$(1+\epsilon)H^*(A) > \mathcal{H}(A);$$

check that for any two adjacent points on $\vec{W}_{III} \cup \vec{W}_{II} \cup \vec{W}_I$, $A < B$,

$$(1+\epsilon)H^*(B) \geq H^*(A) \text{ and } (1+\epsilon)V^*(B) \geq V^*(A);$$

and check that for any three adjacent points on $\vec{W}_{III} \cup \vec{W}_{II} \cup \vec{W}_I$, $A < B < C$,

$$(1+\epsilon)V^*(B) \geq \alpha V^*(A) + (1-\alpha)V^*(C),$$

where $\alpha = \frac{B-A}{C-A}$ and $\epsilon = 10^{-4}$.

Note that by construction we have that $H^*(W(H^s)) = H^s$ and that the indifference condition (36) is satisfied for all W's on the grid.

- If not, find $\underline{W} \in [W_{III}, W_{II}]$ such that the four conditions above are satisfied on $[\underline{W}, W^s]$ and stop. Equilibrium exists for all $H_0 \geq \max(H^*(\underline{W}), \underline{H})$.

Leg 4. etc... Repeat the previous leg as long as $H^*(W_{III}) > \underline{H}$.

Facts for the Numerical Procedure

Fact A.4.1. *Suppose that Assumptions 1-2 are satisfied. Let W^* be an equilibrium threshold wealth function and let $\mathcal{E}(W^*)$ be the associated equilibrium. Suppose that the associated $V^*(W)$ function has a kink at $\mathcal{W}(H^s)$. Then, if $H_0 < H^s \frac{(1-\beta)}{\mu(1-\mu\beta)}$, it must be the case that $W_n(H_0) < \mathcal{W}(H^s)$. Moreover, for all $H \in [H^s \frac{(1-\beta)}{\mu(1-\mu\beta)}, H^s)$, $W_n(H) = \mathcal{W}(H^s)$.*

Proof of Fact A.4.1. From the proof of the Theorem (see, in particular, the discussion following Fact A.2.5) we know that $W_n(H)$ must either satisfy the first order condition

$$\mu \frac{dV^*(W_n(H))}{dW} = \frac{1}{H} \tag{A47}$$

or equal $\mathcal{W}(H^s)$ if it is the case that

$$\lim_{W \nearrow \mathcal{W}(H^s)} \mu \frac{dV^*(W)}{dW} \geq \frac{1}{H}.$$

To prove the first statement, we show that if $H_0 < H^s \frac{(1-\beta)}{\mu(1-\mu\beta)}$

$$\lim_{W \nearrow \mathcal{W}(H^s)} \mu \frac{dV^*(W)}{dW} < \frac{1}{H_0}. \tag{A48}$$

To establish this, we make use of three preliminary results.

Claim A.4.1. *If $V^*(W)$ has a kink at $\mathcal{W}(H^s)$, it must be the case that $W_n(H) = \mathcal{W}(H^s)$ on some interval $[\underline{H}, H^s)$, $\underline{H} \in [H_0, H^s)$.*

Proof of Claim A.4.1. If $V^*(W)$ kinks at $\mathcal{W}(H^s)$, then we have that

$$\lim_{W \nearrow \mathcal{W}(H^s)} \mu \frac{dV^*(W)}{dW} - \frac{1}{H^s} > \lim_{W \searrow \mathcal{W}(H^s)} \mu \frac{dV^*(W)}{dW} - \frac{1}{H^s} = 0.$$

By continuity, for H's smaller than but close to H^s we have

$$\lim_{W \nearrow \mathcal{W}(H^s)} \mu \frac{dV^*(W)}{dW} - \frac{1}{H} > 0 > \lim_{W \searrow \mathcal{W}(H^s)} \mu \frac{dV^*(W)}{dW} - \frac{1}{H},$$

implying that $W_n(H) = \mathcal{W}(H^s)$ on some interval $[\underline{H}, H^s)$. ■

Claim A.4.2. *Define the function*

$$\omega(W) = \begin{cases} W^*(H_c(W)) & \text{if } W \in [W^*(H_0), \mathcal{W}(H^s)) \\ W & \text{if } W \in [\mathcal{W}(H^s), \mathcal{W}(1)] \end{cases}.$$

Then, if $W_n(H) = \mathcal{W}(H^s)$ on some interval $[\underline{H}, H^s)$, it is the case that

$$\lim_{W \nearrow \mathcal{W}(H^s)} \omega'(W) = \frac{1}{\beta(1+\mu)}.$$

Proof of Claim A.4.2. Note from Fact A.4.2 below that, since $W_n(H) = \mathcal{W}(H^s)$ on the interval $[\underline{H}, H^s)$, it follows that

$$F(\omega(\omega(W)), \omega(W), W) = 0, \tag{A49}$$

where the function F is defined below. Differentiating (A49) with respect to W, we obtain

$$F_1(\omega(\omega(W)), \omega(W), W)\omega'(\omega(W))\omega'(W) + F_2(\omega(\omega(W)), \omega(W), W)\omega'(W)$$
$$+F_3(\omega(\omega(W)), \omega(W), W) = 0.$$

Using the fact that $\omega(\mathcal{W}(H^s)) = \mathcal{W}(H^s)$, we have that

$$F_1 \left[\lim_{W \nearrow \mathcal{W}(H^s)} \omega'(W) \right]^2 + F_2 \lim_{W \nearrow \mathcal{W}(H^s)} \omega'(W) + F_3 = 0 \tag{A50}$$

where F_i is the partial derivative evaluated at $(\mathcal{W}(H^s), \mathcal{W}(H^s), \mathcal{W}(H^s))$.

We know from the discussion prior to Fact A.4.2 that

$$F_1 = \left(\frac{\mu\overline{\theta}}{1-\mu\beta} \frac{d\mathcal{H}(\mathcal{W}(H^s))}{dW} \right) \left(\frac{\beta - \beta\mu^2}{1-\beta} \right),$$

$$F_2 = \left(\frac{\mu\overline{\theta}}{1-\mu\beta} \frac{d\mathcal{H}(\mathcal{W}(H^s))}{dW} \right) \left(\frac{\mu - \beta + \mu^2\beta - 1}{1-\beta} \right),$$

and

$$F_3 = \left(\frac{\mu\overline{\theta}}{1-\mu\beta} \frac{d\mathcal{H}(\mathcal{W}(H^s))}{dW} \right) \left(\frac{1-\mu}{1-\beta} \right).$$

Thus, (A50) simplifies to

$$\left(\beta - \beta\mu^2\right) \left[\lim_{W \nearrow \mathcal{W}(H^s)} \omega'(W) \right]^2 + \left(\mu - \beta + \mu^2\beta - 1\right) \left[\lim_{W \nearrow \mathcal{W}(H^s)} \omega'(W) \right] + 1 - \mu = 0$$

The relevant root is

$$\lim_{W \nearrow \mathcal{W}(H^s)} \omega'(W) = \frac{1-\mu}{\beta - \beta\mu^2} = \frac{1}{\beta(1+\mu)}.$$

■

Claim A.4.3. *Assuming that $W_n(H) = \mathcal{W}(H^s)$ on some interval $[\underline{H}, H^s)$, the left derivative of the function $V^*(W)$ at $\mathcal{W}(H^s)$ is*

$$\lim_{W \nearrow \mathcal{W}(H^s)} \mu \frac{dV^*(W)}{dW} = \mu^2 \overline{\theta} \frac{d\mathcal{H}(\mathcal{W}(H^s))}{dW} \left(\frac{1}{1-\beta} \right).$$

Proof of Claim A.4.3. For $W \in [W^*(\underline{H}), \mathcal{W}(H^s))$, we have that

$$V^*(W) - C - \frac{u}{1-\beta} = \mu\overline{\theta}(\mathcal{H}(\frac{W-\beta\omega(W)}{1-\beta}) - 1) + \mu\beta\left(V^*(\omega(W)) - C - \frac{u}{1-\beta}\right).$$

Differentiating, taking the limit as $W \nearrow \mathcal{W}(H^s)$, and using the fact that $\omega(\mathcal{W}(H^s)) = \mathcal{W}(H^s)$, we have that

$$\lim_{W\nearrow\mathcal{W}(H^s)} \frac{dV^*(W)}{dW} = \mu\overline{\theta}\frac{d\mathcal{H}(\mathcal{W}(H^s))}{dW}(\frac{1-\beta\lim_{W\nearrow\mathcal{W}(H^s)}\omega'(W)}{1-\beta}) + \lim_{W\nearrow\mathcal{W}(H^s)}\mu\beta\frac{dV^*(W)}{dW}\lim_{W\nearrow\mathcal{W}(H^s)}\omega'(W).$$

It follows from this equation that

$$\lim_{W\nearrow\mathcal{W}(H^s)} \frac{dV^*(W)}{dW} = \mu\overline{\theta}\frac{1}{1-\beta}\frac{d\mathcal{H}(\mathcal{W}(H^s))}{dW}\left(\frac{1-\beta\lim_{W\nearrow\mathcal{W}(H^s)}\omega'(W)}{1-\mu\beta\lim_{W\nearrow\mathcal{W}(H^s)}\omega'(W)}\right). \tag{A51}$$

It follows from Claim A.4.2 that

$$\begin{aligned}\lim_{W\nearrow\mathcal{W}(H^s)} \frac{dV^*(W)}{dW} &= \mu\overline{\theta}\frac{1}{1-\beta}\frac{d\mathcal{H}(\mathcal{W}(H^s))}{dW}\left(\frac{1-\frac{\beta}{\beta(1+\mu)}}{1-\frac{\mu\beta}{\beta(1+\mu)}}\right)\\ &= \mu\overline{\theta}\frac{\mu}{1-\beta}\frac{d\mathcal{H}(\mathcal{W}(H^s))}{dW}.\end{aligned}$$

■

Given Claim A.4.3, to establish (A48), we therefore need to show that

$$\mu^3\overline{\theta}\frac{1}{1-\beta}\frac{d\mathcal{H}(\mathcal{W}(H^s))}{dW} < \frac{1}{H_0}.$$

We know from the definition of H^s that

$$\frac{1}{H^s} = \left(\frac{\mu}{1-\mu\beta}\right)\mu\overline{\theta}\frac{d\mathcal{H}(\mathcal{W}(H^s))}{dW}$$

This implies that

$$\frac{d\mathcal{H}(\mathcal{W}(H^s))}{dW} = \frac{1-\mu\beta}{\mu^2\overline{\theta}H^s}$$

and hence that[7]

$$\mu^3\overline{\theta}\frac{1}{1-\beta}\frac{d\mathcal{H}(\mathcal{W}(H^s))}{dW} = \frac{\mu(1-\mu\beta)}{H^s(1-\beta)}.$$

Thus, since

$$\frac{1}{H_0} > \frac{\mu(1-\mu\beta)}{H^s(1-\beta)},$$

[7] This also implies that

$$\lim_{W\nearrow\mathcal{W}(H^s)} \frac{dV^*(W)}{dW} = \frac{1-\mu\beta}{1-\beta}\frac{1}{H^s}$$

$$\mu^3\overline{\theta}\frac{1}{1-\beta}\frac{d\mathcal{H}(\mathcal{W}(H^s))}{dW} < \frac{1}{H_0}, \tag{A52}$$

as required.

For the second statement, note that following the same argument used to establish (A48), if $H > H^s\frac{(1-\beta)}{\mu(1-\mu\beta)}$, then it must be the case that

$$\lim_{W \nearrow \mathcal{W}(H^s)} \mu\frac{dV^*(W)}{dW} > \frac{1}{H}. \tag{A53}$$

This implies that for such H, $W_n(H) = \mathcal{W}(H^s)$. ■

Our next result concerns the function $F(W_1, W_2, W_3)$ which is defined as follows:

$$F(W_1,W_2,W_3) \equiv \frac{\beta W_2 - W_3 - (\beta\mathcal{W}(H^s)-W_2)}{\mathcal{H}(\frac{W_3-\beta W_t}{1-\beta})} + \mu\overline{\theta}(\mathcal{H}(\frac{W_3-\beta W_2}{1-\beta})-1) + \mu\beta\left(V^*(\mathcal{W}(H^s)) - C - \frac{u}{1-\beta}\right)$$
$$-\left[\mu\overline{\theta}(\mathcal{H}(\frac{W_2-\beta W_1}{1-\beta})-1) + \mu\beta\left(\begin{array}{c}\frac{\beta W_1 - W_2 - (\beta\mathcal{W}(H^s)-W_1)}{\mathcal{H}(\frac{W_t-\beta W_1}{1-\beta})} \\ +\mu\overline{\theta}(\mathcal{H}(\frac{W_2-\beta W_1}{1-\beta})-1) + \mu\beta\left(V^*(\mathcal{W}(H^s)) - C - \frac{u}{1-\beta}\right)\end{array}\right)\right].$$

This function is well defined and differentiable on the domain $[\mathcal{W}(H^s)-\epsilon, \mathcal{W}(H^s)]^3$ for ϵ sufficiently small. Moreover, the functions' partial derivatives are

$$F_1(W_1,W_2,W_3) = \mu\overline{\theta}\frac{\beta}{1-\beta}\frac{d\mathcal{H}(\frac{W_2-\beta W_1}{1-\beta})}{dW} + \mu^2\overline{\theta}\frac{\beta^2}{1-\beta}\frac{d\mathcal{H}(\frac{W_2-\beta W_1}{1-\beta})}{dW}$$
$$-\mu\beta\left(\frac{(\beta+1)\,\mathcal{H}(\frac{W_2-\beta W_1}{1-\beta}) + \frac{d\mathcal{H}(\frac{W_2-\beta W_1}{1-\beta})}{dW}\frac{\beta}{1-\beta}\left(\beta W_1 - W_2 - (\beta\mathcal{W}(H^s)-W_1)\right)}{\mathcal{H}(\frac{W_2-\beta W_1}{1-\beta})^2}\right),$$

$$F_2(W_1,W_2,W_3) = \frac{(\beta+1)\,\mathcal{H}(\frac{W_3-\beta W_2}{1-\beta}) + \frac{\beta}{1-\beta}\frac{d\mathcal{H}(\frac{W_3-\beta W_2}{1-\beta})}{dW}\left(\beta W_2 - W_3 - (\beta\mathcal{W}(H^s)-W_2)\right)}{\mathcal{H}(\frac{W_3-\beta W_2}{1-\beta})^2}$$
$$-\mu\overline{\theta}\frac{\beta}{1-\beta}\frac{d\mathcal{H}(\frac{W_3-\beta W_2}{1-\beta})}{dW} - \mu\overline{\theta}\frac{1}{1-\beta}\frac{d\mathcal{H}(\frac{W_2-\beta W_1}{1-\beta})}{dW} - \mu^2\overline{\theta}\frac{\beta}{1-\beta}\frac{d\mathcal{H}(\frac{W_2-\beta W_1}{1-\beta})}{dW}$$
$$+\left[\mu\beta\left(\frac{\mathcal{H}(\frac{W_2-\beta W_1}{1-\beta}) + \frac{d\mathcal{H}(\frac{W_2-\beta W_1}{1-\beta})}{dW}\frac{1}{1-\beta}\left(\beta W_1 - W_2 - (\beta\mathcal{W}(H^s)-W_1)\right)}{\mathcal{H}(\frac{W_2-\beta W_1}{1-\beta})^2}\right)\right],$$

and

$$F_3(W_1,W_2,W_3) = -\left(\frac{\mathcal{H}(\frac{W_3-\beta W_2}{1-\beta}) + \frac{d\mathcal{H}(\frac{W_3-\beta W_2}{1-\beta})}{dW}\frac{1}{1-\beta}\left(\beta W_2 - W_3 - (\beta\mathcal{W}(H^s)-W_2)\right)}{\mathcal{H}(\frac{W_3-\beta W_2}{1-\beta})^2}\right)$$
$$+\mu\overline{\theta}\frac{1}{1-\beta}\frac{d\mathcal{H}(\frac{W_3-\beta W_2}{1-\beta})}{dW}.$$

Letting $F_i = F_i(\mathcal{W}(H^s), \mathcal{W}(H^s), \mathcal{W}(H^s))$ and using the fact that

$$\mu^2 \overline{\theta} \frac{d\mathcal{H}(\mathcal{W}(H^s))}{dW} \left(\frac{1}{1-\mu\beta} \right) = \frac{1}{H^s}, \tag{A54}$$

these expressions imply that

$$F_1 = \left(\frac{\mu\overline{\theta}}{1-\mu\beta} \frac{d\mathcal{H}(\mathcal{W}(H^s))}{dW} \right) \left(\frac{\beta - \beta\mu^2}{1-\beta} \right),$$

$$F_2 = \left(\frac{\mu\overline{\theta}}{1-\mu\beta} \frac{d\mathcal{H}(\mathcal{W}(H^s))}{dW} \right) \left(\frac{\mu - \beta + \mu^2\beta - 1}{1-\beta} \right),$$

and that

$$F_3 = \left(\frac{\mu\overline{\theta}}{1-\mu\beta} \frac{d\mathcal{H}(\mathcal{W}(H^s))}{dW} \right) \left(\frac{1-\mu}{1-\beta} \right).$$

Fact A.4.2. *Suppose that Assumptions 1-2 are satisfied. Let W^* be an equilibrium threshold wealth function and let $\mathcal{E}(W^*)$ be the associated equilibrium. Suppose that the associated $V^*(W)$ function has a kink at $\mathcal{W}(H^s)$ and that on the interval $[\underline{H}, H^s)$ we have that $W_n(H) = \mathcal{W}(H^s)$. Define the sequence $\langle W_t(W) \rangle_{t=0}^{\infty}$ inductively by $W_0 = W$ and $W_t = W^*(H_c(W_{t-1}))$. Then, if $W \in [W^*(\underline{H}), \mathcal{W}(H^s))$, it is the case that for all $t > 1$*

$$F(W_{t+1}(W), W_t(W), W_{t-1}(W)) = 0.$$

Proof of Fact A.4.2. For all t, the indifference condition implies that

$$V^*(W_t(W)) = \frac{1-\mu\beta}{1-\beta}\underline{u} + \mathcal{P}(H_t, \mathcal{W}(H^s), W_t(W)) + \mu\overline{\theta}(H_t - 1) + \mu\beta(V^*(\mathcal{W}(H^s)) - C).$$

where $H_t = H_c(W_{t-1})$. Moreover, from the definition of $V^*(W_t(W))$ in (34), we have that

$$V^*(W_t(W)) - C = \frac{1-\mu\beta}{1-\beta}\underline{u} + \mu\overline{\theta}(H_{t+1} - 1) + \mu\beta\left(V^*(W_{t+1}(W)) - C\right).$$

It follows that for all t, it must be the case that

$$\begin{aligned} & \mathcal{P}(H_t, \mathcal{W}(H^s), W_t(W)) + \mu\overline{\theta}(H_t - 1) + \mu\beta\left(V^*(\mathcal{W}(H^s)) - C\right) \\ = \; & \mu\overline{\theta}(H_{t+1} - 1) + \mu\beta\left(\frac{1-\mu\beta}{1-\beta}\underline{u} + \mathcal{P}(H_{t+1}, \mathcal{W}(H^s), W_{t+1}(W)) + \mu\overline{\theta}(H_{t+1} - 1) + \mu\beta(V^*(\mathcal{W}(H^s)) - C) \right). \end{aligned}$$

Moreover, we know that both $\mathcal{P}(H_t, W_t(W), W_{t-1}(W))$ and $\mathcal{P}(H_{t+1}, W_{t+1}(W), W_t(W))$ are equal to C and so we can write this equality as

$$\begin{aligned} & \mathcal{P}(H_t, \mathcal{W}(H^s), W_t(W)) - \mathcal{P}(H_t, W_t(W), W_{t-1}(W)) + \mu\overline{\theta}(H_t - 1) + \mu\beta\left(V^*(\mathcal{W}(H^s)) - C - \frac{\underline{u}}{1-\beta} \right) \\ = \; & \mu\overline{\theta}(H_{t+1} - 1) + \mu\beta \left(\begin{array}{c} \mathcal{P}(H_{t+1}, \mathcal{W}(H^s), W_{t+1}(W)) - \mathcal{P}(H_{t+1}, W_{t+1}(W), W_t(W)) \\ + \mu\overline{\theta}(H_{t+1} - 1) + \mu\beta\left(V^*(\mathcal{W}(H^s)) - C - \frac{\underline{u}}{1-\beta} \right) \end{array} \right). \end{aligned}$$

Using the pricing equation (23), we can rewrite this equality as

$$\begin{aligned}
&\frac{\beta W_t(W) - W_{t-1}(W) - (\beta \mathcal{W}(H^s) - W_t(W))}{\mathcal{H}(\frac{W_{t-1}(W)-\beta W_t(W)}{1-\beta})} + \mu\overline{\theta}(\mathcal{H}(\frac{W_{t-1}(W) - \beta W_t(W)}{1-\beta}) - 1) \\
&+\mu\beta\left(V^*(\mathcal{W}(H^s)) - C - \frac{\underline{u}}{1-\beta}\right) \\
= \; & \mu\overline{\theta}(\mathcal{H}(\frac{W_t(W) - \beta W_{t+1}(W)}{1-\beta}) - 1) \\
&+\mu\beta\left(\begin{array}{c} \frac{\beta W_{t+1}(W) - W_t(W) - (\beta \mathcal{W}(H^s) - W_{t+1}(W))}{\mathcal{H}(\frac{W_t(W)-\beta W_{t+1}(W)}{1-\beta})} + \mu\overline{\theta}(\mathcal{H}(\frac{W_t(W)-\beta W_{t+1}(W)}{1-\beta}) - 1) \\ +\mu\beta\left(V^*(\mathcal{W}(H^s)) - C - \frac{\underline{u}}{1-\beta}\right) \end{array}\right).
\end{aligned}$$

This implies that

$$F(W_{t+1}(W), W_t(W), W_{t-1}(W)) = 0,$$

as required. ■

Fact A.4.3. *Suppose that Assumptions 1-2 are satisfied. Let W^* be an equilibrium threshold wealth function and let $\mathcal{E}(W^*)$ be the associated equilibrium. Then, if $H \in [H_0, H^s)$, it is the case that*

$$W^*(H_c(W^*(H))) < W_n(H).$$

Proof of Fact A.4.3. We begin by showing that

$$\mathcal{P}(H, W^*(H_c(W^*(H))), W^*(H)) > C. \tag{A55}$$

Note that since $\mathcal{P}(H_c(W), W^*(H_c(W)), W) = C$, $H < H_c(W^*(H))$, and $\mathcal{P}$ is a continuous function, it is sufficient to show that

$$\frac{\partial \mathcal{P}(H, W', W)}{\partial H} < 0$$

at any solution of the equation $\mathcal{P}(H, W', W) = C$. Indeed, if the above inequality holds, for any W' and W, there cannot be more than one level of the housing stock at which $\mathcal{P}(H, W', W) = C$. To prove the inequality note from (23), that

$$\frac{\partial \mathcal{P}(H, W', W)}{\partial H} = -\overline{\theta} + S'(H) - \frac{(W - \beta W')}{H^2}.$$

Moreover, at a solution

$$-\frac{(W - \beta W')}{H} = \left[(1-H)\overline{\theta} + S(H) - C(1-\beta) - \underline{u}\right].$$

Thus, at a solution

$$\begin{aligned}\frac{\partial \mathcal{P}(H,W',W)}{\partial H} &= -\overline{\theta}+S'(H)+\frac{\left[(1-H)\overline{\theta}+S\left(H\right)-C(1-\beta)-\underline{u}\right]}{H}\\ &= -\left[\frac{C(1-\beta)+\underline{u}-(1-2H)\overline{\theta}-HS'(H)-S\left(H\right)}{H}\right]\\ &= -\left[\frac{\underline{u}+H\overline{\theta}-\big((1-H)\overline{\theta}+S(H)+HS'(H)-C(1-\beta)\big)}{H}\right]<0.\end{aligned}$$

where the last inequality follows from Assumptions 1(i) and 2.

By definition, we know that

$$\mathcal{P}(H,W_n(H),W^*(H))\leq C.$$

Thus, given (A55), $W_n(H)$ must be larger than $W^*(H_c(W^*(H)))$ as required. ■