July 2014

# On-line Appendix for "Bureaucrats, Voters, and Public Investment"


Levon Barseghyan
Department of Economics
Cornell University
Ithaca NY 14853
lb247@cornell.edu

Stephen Coate
Department of Economics
Cornell University
Ithaca NY 14853
sc163@cornell.edu

# 1 Proof of Proposition 1

The strategy of the proof is to first derive the bureaucrat value function implied by the strategy

$$g'(g) = \begin{cases} 2g^* - (1-\delta)g & \text{if } g \leq g^*/(1-\delta) \\ (1-\delta)g & \text{if } g > g^*/(1-\delta) \end{cases}. \quad (1)$$

We then show that given this value function, the bureaucrat is always better off choosing according to the strategy (1). It follows that there exists a Romer-Rosenthal equilibrium with this value function for the bureaucrat, the value function

$$V(g) = \left(\frac{b_0}{1-\beta(1-\delta)}\right) g - \left(\frac{b_1}{1-\beta(1-\delta)^2}\right) g^2 \quad (2)$$

for the voter, and bureaucrat strategy (1).

## 1.1 The bureaucrat's value function

We begin by deriving the bureaucrat value function implied by the strategy (1). As in Section 2.3, let $g^{**}$ denote the level of public good at which $g'(g)$ equals $g^*/(1-\delta)$. This public good level can be shown to equal $g^*(1-2\delta)/(1-\delta)^2$. As argued in the text, if $g$ is less than $g^{**}$, then new investment is such that $g'(g)$ exceeds $g^*/(1-\delta)$ in which case there is no investment in the subsequent period (see Figure 3). By contrast, if $g$ lies in the interval $(g^{**}, g^*/(1-\delta)]$, then new investment is such that $g'(g)$ is less than $g^*/(1-\delta)$ implying there is investment in the subsequent period. Moreover, $g'(g)$ exceeds $g^{**}$, so that there will be investment in the following and all future periods. To see that $g'(g)$ exceeds $g^{**}$, note first that $g'(g)$ is greater than or equal to $g'(g^*/(1-\delta)) = g^*$. Then observe that $g^*$ exceeds $g^{**} = g^*(1-2\delta)/(1-\delta)^2$.

For the purposes of deriving the bureaucrat's value function, there are three cases to consider depending on the current level of the public good $g$. This level could be less than $g^{**}$, between $g^{**}$ and $g^*/(1-\delta)$, or larger than $g^*/(1-\delta)$. The second case is easiest because thereafter the public good level remains between $g^{**}$ and $g^*/(1-\delta)$ and investment is always positive. In particular, letting period 0 denote the current period, period 1 the subsequent period, and so on, we have that $g_0 = g$ and that for all periods $t$ beyond 0, $g_t = 2g^* - (1-\delta)g_{t-1}$. This is a first order linear difference equation and by a standard formula, the public good level in period $t$ will be

$$g_t = (\delta - 1)^t (g - \frac{2g^*}{2-\delta}) + \frac{2g^*}{2-\delta}. \tag{3}$$

Substituting this formula into

$$U(g_0) = \sum_{t=0}^{\infty} \beta^t g_t, \tag{4}$$

we conclude that on the interval $[g^{**}, g^*/(1-\delta)]$, the bureaucrat's value function is

$$U(g) = \frac{g}{1+\beta(1-\delta)} + \frac{\beta 2g^*}{(1-\beta)(1+\beta(1-\delta))}. \tag{5}$$

Notice that the bureaucrat's value function is increasing linearly in the current level of the public good in this range.

Next consider the case in which the current level of the public good is larger than $g^*/(1-\delta)$. In this case, the public good must depreciate before investment is undertaken. Define $\tau(g)$ to be the smallest integer such that $(1-\delta)^\tau g$ is less than or equal to $g^*/(1-\delta)$. Intuitively, starting in period 0 with $g$ units, after $\tau(g)$ periods of no investment the public good level will have dropped below $g^*/(1-\delta)$. Investment will therefore commence in period $\tau(g)$ when the initial level of the public good has fallen to $(1-\delta)^{\tau(g)} g$. Since $(1-\delta)^{\tau(g)} g$ must lie in the interval $[g^{**}, g^*/(1-\delta)]$, from then on investment will be positive. It follows that for periods $t$ up to and including $\tau(g)$, we have $g_t = (1-\delta)^t g$ and for all subsequent periods $t$, we have $g_t = 2g^* - (1-\delta) g_{t-1}$. This implies that when $g$ exceeds $g^*/(1-\delta)$, the bureaucrat's value function is

$$U(g) = \sum_{t=0}^{\tau(g)-1} \beta^t (1-\delta)^t g + \beta^{\tau(g)} \left( \frac{(1-\delta)^{\tau(g)} g}{1+\beta(1-\delta)} + \frac{\beta 2g^*}{(1-\beta)(1+\beta(1-\delta))} \right). \tag{6}$$

In this case, the value function is also increasing in the current level of the public good, but because of the effect of $g$ on the number of periods of non-investment, the impact is no longer linear.

Finally, consider the case in which the current level of the public good is less than $g^{**}$. In this case, after investment in period 0, the initial level of the public good in period 1 is $2g^* - (1-\delta)g$ which exceeds $g^*/(1-\delta)$. Period 0 will therefore be followed by a spell with no investment. Investment will not recommence until period $\tau(2g^* - (1-\delta)g) + 1$ when the initial level of the public good has fallen below $g^*/(1-\delta)$. From then on investment will be positive. Thus, we have that for $g \in [0, g^{**})$

$$U(g) = g + \sum_{t=1}^{\tau(2g^*-(1-\delta)g)} \beta^t (1-\delta)^{t-1} (2g^* - (1-\delta)g)$$
$$+\beta^{\tau(2g^*-(1-\delta)g)+1} \left( \frac{(1-\delta)^{\tau(2g^*-(1-\delta)g)}(2g^*-(1-\delta)g)}{1+\beta(1-\delta)} + \frac{\beta 2g^*}{(1-\beta)(1+\beta(1-\delta))} \right). \tag{7}$$

This case is much more complicated than the previous two with respect to how the current level of the public good impacts the bureaucrat's value function. This reflects the fact that a lower current level of public good leads to a higher initial investment level.

Combining the three cases, we have the following expression for the bureaucrat's value function:

$$U(g) = \begin{cases} g + \sum_{t=1}^{\tau(2g^*-(1-\delta)g)} \beta^t (1-\delta)^{t-1} (2g^* - (1-\delta)g) \\ +\beta^{\tau(2g^*-(1-\delta)g)+1} \left( \frac{(1-\delta)^{\tau(2g^*-(1-\delta)g)}(2g^*-(1-\delta)g)}{1+\beta(1-\delta)} + \frac{\beta 2g^*}{(1-\beta)(1+\beta(1-\delta))} \right) & \text{if } g \in [0, g^{**}) \\ \frac{g}{1+\beta(1-\delta)} + \frac{\beta 2g^*}{(1-\beta)(1+\beta(1-\delta))} & \text{if } g \in [g^{**}, g^*/(1-\delta)] \\ \sum_{t=0}^{\tau(g)-1} \beta^t (1-\delta)^t g + \beta^{\tau(g)} \left( \frac{(1-\delta)^{\tau(g)} g}{1+\beta(1-\delta)} + \frac{\beta 2g^*}{(1-\beta)(1+\beta(1-\delta))} \right) & \text{if } g > g^*/(1-\delta) \end{cases} \tag{8}$$

With this expression in hand, we can now establish the following useful result:

**Lemma 1** *The bureaucrat's value function as defined in (8) is continuous. On the interval* $[0, g^{**})$*, depending on the parameters* $\beta$ *and* $\delta$*, the value function is either increasing, decreasing, or first decreasing and then increasing. On the interval* $[g^{**}, \infty)$*, the value function is increasing.*

**Proof of Lemma 1** We begin by analyzing the behavior of the value function (8) on the interval $[0, g^{**})$. From (8), for all $g$ in this interval, we have that

$$U(g) = g + \sum_{t=1}^{\tau(2g^*-(1-\delta)g)} \beta^t (1-\delta)^{t-1} (2g^* - (1-\delta)g)$$
$$+\beta^{\tau(2g^*-(1-\delta)g)+1} \left( \frac{(1-\delta)^{\tau(2g^*-(1-\delta)g)}(2g^*-(1-\delta)g)}{1+\beta(1-\delta)} + \frac{\beta 2g^*}{(1-\beta)(1+\beta(1-\delta))} \right)$$

We will show that, on the interval $[0, g^{**})$, $U(g)$ is continuous and either increasing, decreasing, or first decreasing and then increasing.

Define the function

$$\Psi(g, \tau) = g + \sum_{t=1}^{\tau} \beta^t (1-\delta)^{t-1} (2g^* - (1-\delta)g)$$
$$+\beta^{\tau+1} \left( \frac{(1-\delta)^{\tau}(2g^*-(1-\delta)g)}{1+\beta(1-\delta)} + \frac{\beta 2g^*}{(1-\beta)(1+\beta(1-\delta))} \right),$$

and observe that $U(g) = \Psi(g, \tau(2g^* - (1-\delta)g))$. It is the case that

$$\tau(2g^* - (1-\delta)g) = \tau \ \text{ if } 2g^* - (1-\delta)g \in (\frac{g^*}{(1-\delta)^{\tau}}, \frac{g^*}{(1-\delta)^{\tau+1}}],$$

which implies that

$$\tau(2g^* - (1-\delta)g) = \tau \quad \text{if } g \in [\left(\frac{2(1-\delta)^{\tau+1}-1}{(1-\delta)^{\tau+2}}\right) g^*, \left(\frac{2(1-\delta)^{\tau}-1}{(1-\delta)^{\tau+1}}\right) g^*).$$

It follows that $\tau(2g^* - (1-\delta)g)$ is a step function on $[0, g^{**})$, constant and equal to $\tau$ on the interval $[\left(\frac{2(1-\delta)^{\tau+1}-1}{(1-\delta)^{\tau+2}}\right) g^*, \left(\frac{2(1-\delta)^{\tau}-1}{(1-\delta)^{\tau+1}}\right) g^*)$ and dropping down to $\tau - 1$ at the point $\left(\frac{2(1-\delta)^{\tau}-1}{(1-\delta)^{\tau+1}}\right) g^*$. Since $g^{**} = g^*(1-2\delta)/(1-\delta)^2$, we know that $\tau(2g^* - (1-\delta)g) = 1$ if $g \in [\left(\frac{2(1-\delta)^{2}-1}{(1-\delta)^{3}}\right) g^*, g^{**})$, so that at the end of the interval $[0, g^{**})$, $\tau(2g^* - (1-\delta)g)$ drops down to 1.

From this discussion, it is immediate that $U(g)$ is continuous on the interval $[\left(\frac{2(1-\delta)^{\tau+1}-1}{(1-\delta)^{\tau+2}}\right) g^*, \left(\frac{2(1-\delta)^{\tau}-1}{(1-\delta)^{\tau+1}}\right) g^*)$. We can also show that

$$\Psi(\left(\frac{2(1-\delta)^{\tau}-1}{(1-\delta)^{\tau+1}}\right) g^*, \tau) = \Psi(\left(\frac{2(1-\delta)^{\tau}-1}{(1-\delta)^{\tau+1}}\right) g^*, \tau - 1),$$

which establishes continuity at the points at which $\tau(2g^* - (1-\delta)g)$ jumps down. To prove the equality, we need to show that

$$\begin{array}{c} \sum_{t=1}^{\tau} \beta^t (1-\delta)^{t-1} \left(2g^* - \left(\frac{2(1-\delta)^{\tau}-1}{(1-\delta)^{\tau}}\right) g^*\right) \\ +\beta^{\tau+1} \left(\frac{(1-\delta)^{\tau}\left(2g^* - \left(\frac{2(1-\delta)^{\tau}-1}{(1-\delta)^{\tau}}\right) g^*\right)}{1+\beta(1-\delta)} + \frac{\beta 2g^*}{(1-\beta)(1+\beta(1-\delta))}\right) \\ = \sum_{t=1}^{\tau-1} \beta^t (1-\delta)^{t-1} \left(2g^* - \left(\frac{2(1-\delta)^{\tau}-1}{(1-\delta)^{\tau}}\right) g^*\right) \\ +\beta^{\tau} \left(\frac{(1-\delta)^{\tau-1}\left(2g^* - \left(\frac{2(1-\delta)^{\tau}-1}{(1-\delta)^{\tau}}\right) g^*\right)}{1+\beta(1-\delta)} + \frac{\beta 2g^*}{(1-\beta)(1+\beta(1-\delta))}\right) \end{array}$$

Eliminating the common terms from both sides of the equality, this amounts to showing

$$1 + \beta\left(\frac{(1-\delta)}{1+\beta(1-\delta)} + \frac{\beta(1-\delta)2}{(1-\beta)(1+\beta(1-\delta))}\right) = \frac{1}{1+\beta(1-\delta)} + \frac{\beta(1-\delta)2}{(1-\beta)(1+\beta(1-\delta))}$$

This requires that

$$1 - \left(\frac{1-\beta(1-\delta)}{1+\beta(1-\delta)}\right) = \frac{\beta(1-\delta)2}{(1+\beta(1-\delta))},$$

which is true.

We now show that $U(g)$ is either increasing, decreasing, or first decreasing and then increasing on the interval $[0, g^{**})$. Differentiating with respect to $g$, we have that

$$\frac{\partial \Psi(g,\tau)}{\partial g} = 1 - \sum_{t=1}^{\tau} \beta^t (1-\delta)^t - \frac{\beta^{\tau+1}(1-\delta)^{\tau+1}}{1+\beta(1-\delta)}.$$

This derivative, which is independent of $g$, could be positive or negative depending on $\tau$ and the values of $\beta$ and $\delta$. However, observe that for all $\tau$ $\partial\Psi(g,\tau)/\partial g > \partial\Psi(g,\tau+1)/\partial g$, implying that if the derivative is negative for some $\tau$ it will be negative for all higher $\tau$.

We can now establish the claim. Suppose first that $\partial\Psi(g,\tau(2g^*))/\partial g > 0$. Since $\tau(2g^* - (1-\delta)g)$ is maximized at $g = 0$, we know that for all $g$ in the interval $[0, g^{**})$, $\partial\Psi(g,\tau(2g^* - (1-\delta)g))/\partial g > 0$. It follows that $U(g)$ is increasing on the interval $[0, g^{**})$. Next suppose that $\partial\Psi(g,1)/\partial g < 0$. Since $\tau(2g^* - (1-\delta)g)$ is at least as big as 1 for all $g$ in the interval $[0, g^{**})$, we know that for all $g$, $\partial\Psi(g,\tau(2g^* - (1-\delta)g))/\partial g < 0$. It follows that $U(g)$ is decreasing on the interval $[0, g^{**})$. Finally, suppose that $\partial\Psi(g,\tau(2g^*))/\partial g < 0 < \partial\Psi(g,1)/\partial g$. Then, $U(g)$ is first decreasing on the interval $[0, g^{**})$ and then increasing.

We next turn to the behavior of the value function (8) on the interval $[g^{**}, g^*/(1-\delta))$. From (8), for all $g$ in this interval, we have that

$$U(g) = \frac{g}{1+\beta(1-\delta)} + \frac{\beta 2g^*}{(1-\beta)\left(1+\beta(1-\delta)\right)}.$$

It is clear that $U(g)$ is continuous on the interval $(g^{**}, g^*/(1-\delta))$. Moreover,

$$\frac{dU(g)}{dg} = \frac{1}{1+\beta(1-\delta)} > 0,$$

and thus $U(g)$ is increasing on this interval as claimed. Note also that it is the case that

$$\frac{g^{**}}{1+\beta(1-\delta)} + \frac{\beta 2g^*}{(1-\beta)\left(1+\beta(1-\delta)\right)} = \Psi(g^{**},1),$$

which implies that the function $U(g)$ is continuous at the switch point $g^{**}$. To verify the equality note that with a little work it can be shown that

$$\begin{aligned} &\tfrac{g^{**}}{1+\beta(1-\delta)} + \tfrac{\beta 2g^*}{(1-\beta)(1+\beta(1-\delta))} - \Psi(g^{**},1) \\ &= \tfrac{\beta 2g^*(1-\beta^2)}{(1-\beta)(1+\beta(1-\delta))} + g^{**}\left(\tfrac{2\beta^2(1-\delta)^2}{1+\beta(1-\delta)}\right) - 2g^*\left(\tfrac{\beta+2\beta^2(1-\delta)}{1+\beta(1-\delta)}\right). \end{aligned}$$

Using the fact that $g^{**} = g^*\left(1-2\delta\right)/(1-\delta)^2$, we can write the right hand side as

$$\frac{\beta 2g^*(1+\beta)}{(1+\beta(1-\delta))} + 2g^*\left(\frac{(1-2\delta)\beta^2}{1+\beta(1-\delta)}\right) - 2g^*\left(\frac{\beta+2\beta^2(1-\delta)}{1+\beta(1-\delta)}\right),$$

which simplifies to

$$\frac{\beta 2g^*}{(1+\beta(1-\delta))}\left[\beta + (1-2\delta)\beta - 2\beta(1-\delta)\right] = 0.$$

Finally, we turn to the behavior of the value function (8) on the interval $[g^*/(1-\delta), \infty)$. From (8), for all $g$ in this interval, we have that

$$U(g) = \sum_{t=0}^{\tau(g)-1} \beta^t(1-\delta)^t g + \beta^{\tau(g)}\left(\frac{(1-\delta)^{\tau(g)}g}{1+\beta(1-\delta)} + \frac{\beta 2g^*}{(1-\beta)\left(1+\beta(1-\delta)\right)}\right).$$

Define the function

$$\Phi(g,\tau)=\sum_{t=0}^{\tau-1}\beta^t(1-\delta)^t g+\beta^\tau\left(\frac{(1-\delta)^\tau g}{1+\beta(1-\delta)}+\frac{2g^*}{(1-\beta)\left(1+\beta(1-\delta)\right)}\right),$$

and observe that we may write $U(g)=\Phi(g,\tau(g))$. We know that

$$\tau(g)=\tau \text{ if } g\in(\frac{g^*}{(1-\delta)^\tau},\frac{g^*}{(1-\delta)^{\tau+1}}].$$

Thus, $\tau(g)$ is constant and equal to $\tau$ on the interval $(\frac{g^*}{(1-\delta)^\tau},\frac{g^*}{(1-\delta)^{\tau+1}})$, jumping up to $\tau+1$ at $\frac{g^*}{(1-\delta)^{\tau+1}}$. Note that $\tau(g)$ starts out equal to 1 on the interval $[\frac{g^*}{1-\delta},\frac{g^*}{(1-\delta)^2})$.

It is immediate that $U(g)$ is continuous on the interval $(\frac{g^*}{(1-\delta)^\tau},\frac{g^*}{(1-\delta)^{\tau+1}})$. We can also show that

$$\Phi(\frac{g^*}{(1-\delta)^{\tau+1}};\tau)=\Phi(\frac{g^*}{(1-\delta)^{\tau+1}};\tau+1),$$

which implies that the function $U(g)$ is continuous at the points at which $\tau(g)$ jumps up. To prove the equality, we need to show that

$$\begin{aligned}&\sum_{t=0}^{\tau-1}\beta^t(1-\delta)^t\frac{g^*}{(1-\delta)^{\tau+1}}+\beta^\tau\left(\frac{(1-\delta)^\tau\frac{g^*}{(1-\delta)^{\tau+1}}}{1+\beta(1-\delta)}+\frac{\beta 2g^*}{(1-\beta)\left(1+\beta(1-\delta)\right)}\right)\\ =\;&\sum_{t=0}^{\tau}\beta^t(1-\delta)^t\frac{g^*}{(1-\delta)^{\tau+1}}+\beta^{\tau+1}\left(\frac{(1-\delta)^{\tau+1}\frac{g^*}{(1-\delta)^{\tau+1}}}{1+\beta(1-\delta)}+\frac{\beta 2g^*}{(1-\beta)\left(1+\beta(1-\delta)\right)}\right)\end{aligned}$$

Eliminating the common terms from both sides of the equality, this amounts to showing

$$1+\beta\left(\frac{(1-\delta)}{1+\beta(1-\delta)}+\frac{\beta(1-\delta)2}{(1-\beta)\left(1+\beta(1-\delta)\right)}\right)=\frac{1}{1+\beta(1-\delta)}+\frac{\beta(1-\delta)2}{(1-\beta)\left(1+\beta(1-\delta)\right)}$$

This requires that

$$1-\left(\frac{1-\beta(1-\delta)}{1+\beta(1-\delta)}\right)=\frac{\beta(1-\delta)2}{(1+\beta(1-\delta))},$$

which is true.

To see that $U(g)$ is increasing on the interval $[g^*/(1-\delta),\infty)$, note that differentiating with respect to $g$, we have that

$$\frac{\partial\Phi(g,\tau)}{\partial g}=\sum_{t=1}^{\tau-1}\beta^t(1-\delta)^t+\frac{\beta^\tau(1-\delta)^\tau}{1+\beta(1-\delta)}>0.$$

Finally, note that $U(g)$ is continuous at $g^*/(1-\delta)$, since

$$\Phi(g^*/(1-\delta),1)=\frac{g^*}{(1-\delta)\left(1+\beta(1-\delta)\right)}+\frac{\beta 2g^*}{(1-\beta)\left(1+\beta(1-\delta)\right)}.$$

This completes the proof of Lemma 1. ■

## 1.2 Optimality of the strategy (1)

With the expression for the bureaucrat's value function presented in equation (8) and Lemma 1, it is now possible to show that the bureaucrat always wants to implement the maximum level of investment the citizen will support. While somewhat involved, the proof is conceptually straightforward. As noted in the text, we need to show that for any $g$ less than $g^*/(1-\delta)$, $U(g'(g))$ is at least as large as $U(g')$ for any $g'$ in the feasible set $[(1-\delta)g, g'(g)]$. By Lemma 1, if the feasible set $[(1-\delta)g, g'(g)]$ is a subset of $[g^{**}, \infty)$, the result is immediate because $U(g')$ is increasing. If the feasible set intersects the interval $[0, g^{**})$, matters are more complicated, because $U(g')$ may be decreasing on some part of the interval $[(1-\delta)g, g^{**})$. However, Lemma 1 tells us that it is only necessary to rule out the possibility that $U((1-\delta)g)$ exceeds $U(g'(g))$. Intuitively, if the bureaucrat does want to deviate from the equilibrium strategy it will be by not investing at all, rather than investing some smaller amount. This limits the comparisons that need to be made.

Turning to the details, we need to show that given the value functions (2) and (8), the proposed policy function (1) solves the bureaucrat's problem

$$\max_{g'} \left\{ \begin{array}{c} g + \beta U(g') \\ \\ s.t.\ \beta\left[V(g') - V(g(1-\delta))\right] \geq c(g' - g(1-\delta)) \ \ \&\ g' \geq (1-\delta)g \end{array} \right\}.$$

This amounts to showing that for all $g \leq g^*/(1-\delta)$,

$$2g^* - (1-\delta)g \in \arg\max\{U(g') : g' \in [(1-\delta)g, 2g^* - (1-\delta)g]\} \tag{9}$$

Let $g \leq g^*/(1-\delta)$. Suppose first that $(1-\delta)g \geq g^{**}$. Then, since $2g^* - (1-\delta)g \leq g^*/(1-\delta)$ we know that the interval $[(1-\delta)g, 2g^* - (1-\delta)g]$ is a subset of the interval $[g^{**}, g^*/(1-\delta)]$. Thus, from Lemma 1 we know that $U(g')$ is increasing on the interval $[(1-\delta)g, 2g^* - (1-\delta)g]$. It follows immediately that (9) is satisfied.

If $(1-\delta)g < g^{**}$ then we know that the interval $[(1-\delta)g, 2g^* - (1-\delta)g]$ overlaps with the interval $[0, g^{**})$. If $U(g')$ is increasing on the interval $[(1-\delta)g, g^{**})$ then, by Lemma 1 it is increasing on the interval $[(1-\delta)g, 2g^* - (1-\delta)g]$ and (9) is satisfied. If $U(g')$ is not increasing on the interval $[(1-\delta)g, g^{**})$, then Lemma A.l tells us that there exists some $\widetilde{g} \in ((1-\delta)g, g^{**})$ such that $U(g')$ is decreasing on $[(1-\delta)g, \widetilde{g})$ and increasing thereafter. It follows immediately that either (9) is satisfied or it must be the case that

$$(1-\delta)g = \arg\max\{U(g') : g' \in [(1-\delta)g, 2g^* - (1-\delta)g]\}.$$

To prove the proposition, we must therefore establish that when $(1-\delta)g < g^{**}$, it is the case that

$$U(2g^* - (1-\delta)g) \geq U((1-\delta)g). \tag{10}$$

Intuitively, we just need to check that the equilibrium strategy dominates the strategy of investing nothing.

There are two possibilities to consider depending on the value of the initial level of public good $g$. These possibilities are (i) $g \in [g^{**}, \frac{g^{**}}{1-\delta})$ and (ii) $g \in [0, g^{**})$. In case (i), we know that $2g^* - (1-\delta)g \leq g^*/(1-\delta)$, while in case (ii) we know that $2g^* - (1-\delta)g > g^*/(1-\delta)$. From (8), this impacts the expression for the equilibrium payoff $U(2g^* - (1-\delta)g)$. We deal with each case in turn.

### 1.2.1 Case (i) $g \in [g^{**}, \frac{g^{**}}{1-\delta})$.

Since $(1-\delta)g < g^{**}$ and $2g^* - (1-\delta)g \leq g^*/(1-\delta)$, it follows from (8) that to establish (10), we need to show that

$$\frac{2g^* - (1-\delta)g}{1+\beta(1-\delta)} + \frac{\beta 2g^*}{(1-\beta)\left(1+\beta(1-\delta)\right)} \tag{11}$$

exceeds

$$\begin{gathered}(1-\delta)g + \sum_{t=1}^{\tau(2g^*-(1-\delta)^2 g)} \beta^t (1-\delta)^{t-1} \left(2g^* - (1-\delta)^2 g\right) \\ +\beta^{\tau(2g^*-(1-\delta)^2 g)+1}\left(\frac{(1-\delta)^{\tau(2g^*-(1-\delta)^2 g)}\left(2g^*-(1-\delta)^2 g\right)}{1+\beta(1-\delta)} + \frac{\beta 2g^*}{(1-\beta)(1+\beta(1-\delta))}\right).\end{gathered} \tag{12}$$

We begin by establishing the following useful fact.

**Claim 1** $\tau(2g^* - (1-\delta)^2 g) = 1$.

Intuitively, this tells us that if the bureaucrat invests nothing, then the investment which follows in the next period will be followed by just one period in which no investment takes place.

**Proof of Claim 1** We know that $\tau(g) = \tau$ if $g \in (\frac{g^*}{(1-\delta)^\tau}, \frac{g^*}{(1-\delta)^{\tau+1}}]$. Thus, it must be the case that

$$2g^* - (1-\delta)^2 g \in (\frac{g^*}{(1-\delta)^{\tau_d}}, \frac{g^*}{(1-\delta)^{\tau_d+1}}], \tag{13}$$

where $\tau_d = \tau(2g^* - (1-\delta)^2 g)$. It follows from (13) that $g \in [g^*\left(\frac{2(1-\delta)^{\tau_d+1}-1}{(1-\delta)^{\tau_d+3}}\right), g^*\left(\frac{2(1-\delta)^{\tau_d}-1}{(1-\delta)^{\tau_d+2}}\right))$. We also know that $g > g^{**}$ which, given the definition of $g^{**}$, implies that $g > g^*\left(\frac{2(1-\delta)-1}{(1-\delta)^2}\right)$. For both these things to be possible, it must be the case that

$$\frac{2(1-\delta)-1}{(1-\delta)} < \frac{2(1-\delta)^{\tau_d}-1}{(1-\delta)^{\tau_d+1}}.$$

Suppose, contrary to the claim, that $\tau_d \geq 2$. Then, it must be the case that

$$\frac{2(1-\delta)^2-1}{(1-\delta)^3} > \frac{2(1-\delta)-1}{(1-\delta)}.$$

The inequality is equivalent to $2\delta(1-\delta)^2 > 1-(1-\delta)^2$, which reduces to $(1-\delta)^2 > 1-\frac{\delta}{2}$. But

$$(1-\delta)^2 - \left(1-\frac{\delta}{2}\right) = \delta^2 - \frac{3}{2}\delta < 0,$$

which is a contradiction. ■

In light of Claim 1, we can write the payoff from not investing (12) more simply as

$$(1-\delta)g + \beta(2g^* - (1-\delta)^2 g) + \beta^2 \left( \frac{(1-\delta)\left(2g^* - (1-\delta)^2 g\right)}{1+\beta(1-\delta)} + \frac{\beta 2g^*}{(1-\beta)\left(1+\beta(1-\delta)\right)} \right) \quad (14)$$

To compare the equilibrium payoff (11) with this, note that if the bureaucrat plays according to the equilibrium, then in the subsequent period the level of public good will be $2g^* - (1-\delta)g$, in the period following that it will be $2g^* - (1-\delta)\left(2g^* - (1-\delta)g\right)$, and in the period following that it will be $2g^* - (1-\delta)\left(2g^* - (1-\delta)\left(2g^* - (1-\delta)g\right)\right)$. Thus, we can write the equilibrium payoff (11) as

$$\begin{aligned} &2g^* - (1-\delta)g + \beta(2g^* - (1-\delta)\left(2g^* - (1-\delta)g\right)) \\ &+\beta^2 \left( \tfrac{2g^* - (1-\delta)(2g^* - (1-\delta)(2g^* - (1-\delta)g))}{1+\beta(1-\delta)} + \tfrac{\beta 2g^*}{(1-\beta)(1+\beta(1-\delta))} \right). \end{aligned} \quad (15)$$

Subtracting (14) from (15), we find that the difference between the equilibrium payoff and the payoff from not investing is

$$2\left(g^* - (1-\delta)g\right) + \beta 2(1-\delta)\left((1-\delta)g - g^*\right) + 2\beta^2 \left( \frac{\delta^2 g^*}{1+\beta(1-\delta)} \right). \quad (16)$$

The right hand side of (16) is decreasing in $g$, and thus it suffices to show that it is non-negative at $g = \frac{g^{**}}{1-\delta} = g^* \left( \frac{2(1-\delta)-1}{(1-\delta)^3} \right)$. But this follows immediately from the fact that

$$\frac{g^*}{1-\delta} - \frac{g^{**}}{1-\delta} = \frac{g^*}{1-\delta}\left(1 - \frac{2(1-\delta)-1}{(1-\delta)^2}\right) = g^* \frac{\delta^2}{(1-\delta)^3} > 0.$$

It follows that (10) is satisfied in Case (i).

#### 1.2.2 Case (ii) $g \in [0, g^{**})$.

Since $(1-\delta)g < g^{**}$ and $2g^* - (1-\delta)g > g^*/(1-\delta)$, it follows from (8) that to establish (10), we need to show that

$$\begin{gathered}\sum_{t=0}^{\tau(2g^*-(1-\delta)g)-1} \beta^t (1-\delta)^t (2g^* - (1-\delta)g) + \\ \beta^{\tau(2g^*-(1-\delta)g)} \left( \frac{(1-\delta)^{\tau(2g^*-(1-\delta)g)}(2g^*-(1-\delta)g)}{1+\beta(1-\delta)} + \frac{\beta 2g^*}{(1-\beta)(1+\beta(1-\delta))} \right),\end{gathered} \tag{17}$$

exceeds

$$\begin{gathered}(1-\delta)g + \sum_{t=1}^{\tau(2g^*-(1-\delta)^2 g)} \beta^t (1-\delta)^{t-1} \left(2g^* - (1-\delta)^2 g\right) \\ +\beta^{\tau(2g^*-(1-\delta)^2 g)+1} \left( \frac{(1-\delta)^{\tau(2g^*-(1-\delta)^2 g)}\left(2g^*-(1-\delta)^2 g\right)}{1+\beta(1-\delta)} + \frac{\beta 2g^*}{(1-\beta)(1+\beta(1-\delta))} \right).\end{gathered} \tag{18}$$

To simplify notation let $\tau_d = \tau(2g^* - (1-\delta)^2 g)$ and let $\tau_e = \tau(2g^* - (1-\delta)g)$. It is clear that $\tau_d$ must be at least as large as $\tau_e$. We can also establish the following useful fact.

**Claim 2** $\tau_d \leq \tau_e + 1$.

Intuitively, this tells us that if the bureaucrat invests nothing, then the increase in public good investment that he can obtain the next period is sufficient to raise the number of subsequent periods in which no investment takes place by at most one.

**Proof of Claim 2** We know that $\tau(g) = \tau$ if $g \in (\frac{g^*}{(1-\delta)^{\tau}}, \frac{g^*}{(1-\delta)^{\tau+1}}]$. Thus, it must be the case that $2g^* - (1-\delta)g \in (\frac{g^*}{(1-\delta)^{\tau_e}}, \frac{g^*}{(1-\delta)^{\tau_e+1}}]$, and that $2g^* - (1-\delta)^2 g \in (\frac{g^*}{(1-\delta)^{\tau_d}}, \frac{g^*}{(1-\delta)^{\tau_d+1}}]$. It follows that

$$g \in [g^* \left( \frac{2(1-\delta)^{\tau_e+1} - 1}{(1-\delta)^{\tau_e+2}} \right), g^* \left( \frac{2(1-\delta)^{\tau_e} - 1}{(1-\delta)^{\tau_e+1}} \right)),$$

and that

$$g \in [g^* \left( \frac{2(1-\delta)^{\tau_d+1} - 1}{(1-\delta)^{\tau_d+3}} \right), g^* \left( \frac{2(1-\delta)^{\tau_d} - 1}{(1-\delta)^{\tau_d+2}} \right)).$$

For this to be possible it must be the case that

$$\frac{2(1-\delta)^{\tau_d} - 1}{(1-\delta)^{\tau_d+1}} > \frac{2(1-\delta)^{\tau_e+1} - 1}{(1-\delta)^{\tau_e+1}}.$$

Now suppose, contrary to the claim, that $\tau_d \geq \tau_e + 2$. Then, it must be the case that

$$\frac{2(1-\delta)^{\tau_e+2} - 1}{(1-\delta)^{\tau_e+3}} > \frac{2(1-\delta)^{\tau_e+1} - 1}{(1-\delta)^{\tau_e+1}}$$

The inequality is equivalent to $2(1-\delta)^{\tau_e+2} - 1 > 2(1-\delta)^{\tau_e+3} - (1-\delta)^2$. This in turn is equivalent to $2\delta(1-\delta)^{\tau_e+2} > 1 - (1-\delta)^2$, which reduces to $(1-\delta)^{\tau_e+2} > 1 - \frac{\delta}{2}$. But

$$(1-\delta)^{\tau_e+2} \leq (1-\delta)^2 < 1 - \frac{\delta}{2},$$

which is a contradiction. ■

It follows from Claim 2 that we need only consider two possibilities: $\tau_d = \tau_e$, and $\tau_d = \tau_e + 1$. We begin with the first. Suppose therefore that $\tau_d = \tau_e = \tau \geq 1$. This implies that

$$g \in [g^* \left( \frac{2(1-\delta)^{\tau+1} - 1}{(1-\delta)^{\tau+3}} \right), g^* \left( \frac{2(1-\delta)^{\tau} - 1}{(1-\delta)^{\tau+1}} \right)]. \tag{19}$$

We can write (17) and (18) more compactly as

$$\begin{gathered} \textstyle\sum_{t=0}^{\tau-1} \beta^t (1-\delta)^t (2g^* - (1-\delta)g) + \\ \beta^{\tau} \left( \frac{(1-\delta)^{\tau}(2g^* - (1-\delta)g)}{1+\beta(1-\delta)} + \frac{\beta 2g^*}{(1-\beta)(1+\beta(1-\delta))} \right), \end{gathered} \tag{20}$$

and

$$\begin{gathered} (1-\delta)g + \textstyle\sum_{t=1}^{\tau} \beta^t (1-\delta)^{t-1} \left(2g^* - (1-\delta)^2 g\right) + \\ \beta^{\tau+1} \left( \frac{(1-\delta)^{\tau}\left(2g^* - (1-\delta)^2 g\right)}{1+\beta(1-\delta)} + \frac{\beta 2g^*}{(1-\beta)(1+\beta(1-\delta))} \right). \end{gathered} \tag{21}$$

Recall that expression (20) is the future payoff from playing the equilibrium strategy and expression (21) that from not investing. In interpreting these expressions, note that period $t = 0$ denotes the period after playing the equilibrium strategy or deviating, period $t = 1$ denotes the second period after playing the equilibrium strategy or deviating, etc.[1] To make the two expressions more easily comparable, note that if the bureaucrat plays according to the equilibrium, then in period $\tau$ (which is $\tau + 1$ periods after playing the equilibrium strategy or deviating), the level of public good will be $(1-\delta)^{\tau} (2g^* - (1-\delta)g)$. Given the definition of $\tau$, investment will take place in period $\tau$ so that in period $\tau+1$, the level of public good will be $2g^* - (1-\delta)^{\tau+1}(2g^* - (1-\delta)g)$. Accordingly, we can rewrite (20) as

$$\begin{gathered} 2g^* - (1-\delta)g + \textstyle\sum_{t=1}^{\tau} \beta^t (1-\delta)^t (2g^* - (1-\delta)g) + \\ \beta^{\tau+1} \left( \frac{2g^* - (1-\delta)^{\tau+1}(2g^* - (1-\delta)g)}{1+\beta(1-\delta)} + \frac{\beta 2g^*}{(1-\beta)(1+\beta(1-\delta))} \right). \end{gathered} \tag{22}$$

Subtracting (21) from (22) yields

$$\left[ 2(g^* - (1-\delta)g) - 2\delta \sum_{t=1}^{\tau} \beta^t (1-\delta)^{t-1} g^* \right] + \beta^{\tau+1} \left( \frac{2(1-\delta)^{t+2} g - 2g^* \left((1-\delta)^t (2-\delta) - 1\right)}{1+\beta(1-\delta)} \right). \tag{23}$$

[1] In the period of playing the equilibrium strategy or deviating, the public good level is just $g$.

Expression (23) is the difference between the future equilibrium payoff and the future payoff from not investing. The first term in the square brackets is the difference in public good levels associated with periods 0 through $\tau$ and the second term the difference associated with the remaining periods. Noting that

$$\sum_{t=1}^{\tau} \beta^t (1-\delta)^t = \frac{\beta(1-\delta)(1-[\beta(1-\delta)]^\tau)}{1-\beta(1-\delta)}, \tag{24}$$

we can write the first term as

$$2(g^* - (1-\delta)g) - 2\delta \sum_{t=1}^{\tau} \beta^t (1-\delta)^{t-1} g^* = 2\left[\left(\frac{1-\beta+\beta\delta\,[\beta(1-\delta)]^\tau}{1-\beta(1-\delta)}\right) g^* - (1-\delta)g\right]. \tag{25}$$

This allows us to rewrite (23) as

$$2\left[\left(\frac{1-\beta+\beta\delta\,[\beta(1-\delta)]^\tau}{1-\beta(1-\delta)}\right) g^* - (1-\delta)g\right] + \beta^{\tau+1}\left(\frac{2(1-\delta)^{t+2}g - 2g^*\left((1-\delta)^t(2-\delta)-1\right)}{1+\beta(1-\delta)}\right). \tag{26}$$

Differentiating (26) with respect to $g$ we obtain

$$-2(1-\delta)\left(\frac{1+\beta(1-\delta)-\beta^{\tau+1}(1-\delta)^{\tau+1}}{1+\beta(1-\delta)}\right) < 0,$$

implying that the difference in payoffs is decreasing in $g$. Thus, it suffices to show that (26) is non-negative at the largest possible $g$ in the interval (19) which is $g^*\left(2(1-\delta)^\tau - 1\right)/(1-\delta)^{\tau+1}$.

We do this in two steps. First, we show that at the largest possible $g$ in the interval (19) deviating from the equilibrium strategy by not investing must reduce the bureaucrat's future payoff over the first $\tau$ future periods. We then show that deviating also reduces payoffs in the remaining periods.

**Claim 3**

$$2g^*\left[\left(\frac{1-\beta+\beta\delta\,[\beta(1-\delta)]^\tau}{1-\beta(1-\delta)}\right) - \left(\frac{2(1-\delta)^\tau-1}{(1-\delta)^\tau}\right)\right] \geq 0.$$

**Proof of Claim 3** We can write

$$\begin{aligned} & 2g^*\left[\left(\frac{1-\beta+\beta\delta\,[\beta(1-\delta)]^\tau}{1-\beta(1-\delta)}\right) - \left(\frac{2(1-\delta)^\tau-1}{(1-\delta)^\tau}\right)\right] \\ = \quad & 2g^*\left[\left(\frac{(1-\beta)\left(1-(1-\delta)^\tau\right)+\beta\delta\left(1-2(1-\delta)^\tau+\beta^\tau(1-\delta)^{2\tau}\right)}{\left(1-\beta(1-\delta)\right)(1-\delta)^\tau}\right)\right] \end{aligned}$$

Define the function

$$\varphi(\beta,\delta;\tau) = (1-\beta)\left(1-(1-\delta)^\tau\right) + \beta\delta\left(1-2(1-\delta)^\tau+\beta^\tau(1-\delta)^{2\tau}\right).$$

Then, it suffices to show that $\varphi(\beta,\delta;\tau) \geq 0$. We do this in two steps: first, we show that $\varphi(\beta,\delta;1) \geq 0$ and then we show that for all $\tau \geq 1$ it is the case that $\varphi(\beta,\delta;\tau+1) \geq \varphi(\beta,\delta;\tau)$.

For the first step, note that

$$\begin{aligned}
\varphi(\beta,\delta;1) &= (1-\beta)\delta + \beta\delta\left(2\delta - 1 + \beta(1-\delta)^2\right) \\
&= \delta\left(1 + \beta^2(1-\delta)^2 - 2\beta(1-\delta)\right).
\end{aligned}$$

Thus, it suffices to show that $\phi(\delta) \geq 0$, where

$$\phi(\delta) = 1 + \beta^2(1-\delta)^2 - 2\beta(1-\delta).$$

Note that

$$\phi'(\delta) = 2\beta(1-\beta(1-\delta)) > 0,$$

and thus

$$\min_{\delta\in[0,1]} \phi(\delta) = \phi(0)$$

Now observe that

$$\phi(0) = 1 + \beta^2 - 2\beta = (1-\beta)^2 \geq 0.$$

For the second step, note that

$$\varphi(\beta,\delta;\tau+1) - \varphi(\beta,\delta;\tau) = (1-\delta)^\tau\delta\left(\beta\left[2\delta - \beta^\tau(1-\delta)^\tau(1-\beta(1-\delta)^2)\right] + 1 - \beta\right)$$

Thus, it suffices to show that $\varsigma(\beta,\tau) \geq 0$ where

$$\varsigma(\beta,\tau) = \beta\left(2\delta - \beta^\tau(1-\delta)^\tau(1-\beta(1-\delta)^2)\right) + 1 - \beta.$$

Since $\varsigma(\beta,\tau)$ is decreasing in $\tau$, it suffices to show that $\varsigma(\beta,1) \geq 0$. Note first that $\varsigma(0,1) = 1$ and

$$\varsigma(1,1) = 2\delta - (1-\delta)(2\delta - \delta^2) \geq 0.$$

Thus, if $\varsigma(\beta,1)$ is minimized at either $\beta = 0$ or $\beta = 1$, we are done. If $\varsigma(\beta,1)$ has a minimum on $(0,1)$, call it $\beta^*$, then we know that

$$\frac{\partial\varsigma(\beta^*,1)}{\partial\beta} = 0 \tag{27}$$

Note that we may write

$$\varsigma(\beta,1) = \beta 2\delta + 1 + \beta^3(1-\delta)^3 - \beta^2(1-\delta) - \beta$$

so that

$$\frac{\partial \varsigma(\beta,1)}{\partial \beta} = 2\delta + 3\beta^2(1-\delta)^3 - 2\beta(1-\delta) - 1$$

Thus, (27) implies that

$$3\beta^{*2}(1-\delta)^3 = 2\beta^*(1-\delta) + 1 - 2\delta$$

which implies that

$$\beta^{*3}(1-\delta)^3 = \frac{2\beta^{*2}(1-\delta) + \beta^*(1-2\delta)}{3}.$$

It follows that

$$\begin{aligned}
\varsigma(\beta^*,1) &= \beta^* 2\delta + 1 + \frac{2\beta^{*2}(1-\delta) + \beta^*(1-2\delta)}{3} - \beta^{*2}(1-\delta) - \beta^* \\
&= 1 - \frac{\beta^{*2}(1-\delta)}{3} - \frac{\beta^* 2}{3}(1-2\delta) \\
&\geq 1 - \frac{\beta^{*2}}{3} - \frac{\beta^* 2}{3} \geq 0.
\end{aligned}$$

This completes the proof of Claim 3. ■

The impact of not investing on payoffs in the remaining periods is captured by the second term of (26). We see that this term is non-negative if

$$g \geq g^* \left( \frac{2(1-\delta)^{t+1} - \left(\delta(1-\delta)^{t+1} + (1-\delta)\right)}{(1-\delta)^{t+3}} \right).$$

Evidently, it will be non-negative at the largest possible $g$ if

$$\frac{2(1-\delta)^t - 1}{(1-\delta)^t} \geq \frac{2(1-\delta)^{t+1} - \left(\delta(1-\delta)^{t+1} + (1-\delta)\right)}{(1-\delta)^{t+2}}.$$

To see that this is true, note that this is equivalent to

$$2(1-\delta)^{t+1} - \left(\delta(1-\delta)^{t+1} + (1-\delta)\right) < 2(1-\delta)^{t+2} - (1-\delta)^2,$$

which is turn equivalent to

$$(1-\delta)^{t+1} < 1 - \delta,$$

which is true. This completes the proof for the case in which $\tau_d = \tau_e$.

Now suppose that $\tau_d = \tau_e + 1$. Letting $\tau_e = \tau$, this implies that

$$g \in [g^* \left( \frac{2(1-\delta)^{\tau+2} - 1}{(1-\delta)^{\tau+4}} \right), g^* \left( \frac{2(1-\delta)^{\tau} - 1}{(1-\delta)^{\tau+1}} \right)]. \tag{28}$$

Rewriting (17) and (18), we need to show that for all $g$ in this interval

$$\begin{gathered}\sum_{t=0}^{\tau-1}\beta^t(1-\delta)^t(2g^*-(1-\delta)g)+\\ \beta^\tau\left(\frac{(1-\delta)^\tau(2g^*-(1-\delta)g)}{1+\beta(1-\delta)}+\frac{\beta 2g^*}{(1-\beta)(1+\beta(1-\delta))}\right),\end{gathered} \tag{29}$$

is at least as large as

$$\begin{gathered}(1-\delta)g+\sum_{t=1}^{\tau+1}\beta^t(1-\delta)^{t-1}\left(2g^*-(1-\delta)^2g\right)+\\ \beta^{\tau+2}\left(\frac{(1-\delta)^{\tau+1}\left(2g^*-(1-\delta)^2g\right)}{1+\beta(1-\delta)}+\frac{\beta 2g^*}{(1-\beta)(1+\beta(1-\delta))}\right).\end{gathered} \tag{30}$$

Once again, expression (29) is the future payoff from playing the equilibrium strategy and expression (30) the future payoff from deviating. To make the two expressions more easily comparable, note that if the bureaucrat plays according to the equilibrium, then in period $\tau$ (which is $\tau+1$ periods after playing the equilibrium strategy or deviating), the level of public good will be $(1-\delta)^\tau(2g^*-(1-\delta)g)$. Given the definition of $\tau$, investment will take place in period $\tau$ so that in period $\tau+1$, the level of public good will be $2g^*-(1-\delta)^{\tau+1}(2g^*-(1-\delta)g)$. Investment will also take place in period $\tau+1$ so that in period $\tau+2$ the level of the public good will be $2g^*-(1-\delta)(2g^*-(1-\delta)^{\tau+1}(2g^*-(1-\delta)g))$. Accordingly, we can rewrite (29) as

$$\begin{gathered}2g^*-(1-\delta)g+\sum_{t=1}^{\tau}\beta^t(1-\delta)^t(2g^*-(1-\delta)g)+\beta^{\tau+1}(2g^*-(1-\delta)^{\tau+1}(2g^*-(1-\delta)g))\\ +\beta^{\tau+2}\left(\frac{\delta 2g^*+(1-\delta)^{\tau+2}(2g^*-(1-\delta)g)}{1+\beta(1-\delta)}+\frac{\beta 2g^*}{(1-\beta)(1+\beta(1-\delta))}\right)\end{gathered} \tag{31}$$

Subtracting (30) from (31) and using (25), yields

$$\begin{gathered}2\left[\left(\frac{1-\beta+\beta\delta[\beta(1-\delta)]^\tau}{1-\beta(1-\delta)}\right)g^*-(1-\delta)g\right]\\ +\beta^{\tau+1}\left[2(1-\delta)^{\tau+2}g+2g^*\left(1-(1-\delta)^\tau(2-\delta)\right)\right]+\beta^{\tau+2}\left(\frac{\delta 2g^*[1-(1-\delta)^{\tau+1}]}{1+\beta(1-\delta)}\right)\end{gathered} \tag{32}$$

This expression is the difference between the future equilibrium payoff and the future payoff from deviating and we need to demonstrate that it is non-negative for all $g$ in the interval (28).

Differentiating (32) it is easy to show that the difference in payoffs is decreasing in $g$. Thus, it suffices to show that (32) is non-negative when evaluated at the largest possible $g$ in the interval (28) which is $g^*\left(\frac{2(1-\delta)^\tau-1}{(1-\delta)^{\tau+1}}\right)$. We have already proved that the first term is non-negative in Claim 3. It is therefore sufficient to show that, combined, the second and third terms are non-negative.

Combining the second and third terms, we can write

$$\begin{aligned}&\beta^{\tau+1}\left[2(1-\delta)^{\tau+2}g+2g^*\left(1-(1-\delta)^{\tau}(2-\delta)\right)\right]+\beta^{\tau+2}\left(\tfrac{\delta 2g^*\left[1-(1-\delta)^{\tau+1}\right]}{1+\beta(1-\delta)}\right)\\&=\beta^{\tau+1}\left(\tfrac{2(1-\delta)^{\tau+2}g(1+\beta(1-\delta))+2g^*\left[1-(1-\delta)^{\tau+1}-(1-\delta)^{\tau}+\beta(1-2(1-\delta)^{\tau+1})\right]}{1+\beta(1-\delta)}\right)\end{aligned} \tag{33}$$

Thus, it is enough to show that

$$(1-\delta)\left(2(1-\delta)^{\tau}-1\right)\left(1+\beta(1-\delta)\right)+\left[1-(1-\delta)^{\tau+1}-(1-\delta)^{\tau}+\beta(1-2(1-\delta)^{\tau+1})\right]\geq 0 \tag{34}$$

Differentiating this expression with respect to $\beta$, we see that it is increasing in $\beta$. Thus, we just need to check that it is non-negative at $\beta=0$. This requires that

$$(1-\delta)\left(2(1-\delta)^{\tau}-1\right)+\left[1-(1-\delta)^{\tau+1}-(1-\delta)^{\tau}\right]\geq 0, \tag{35}$$

which is equivalent to

$$\delta-(1-\delta)^{\tau}\delta\geq 0. \tag{36}$$

This is true. It follows (10) is also satisfied in Case (ii).

We conclude that there exists a Romer-Rosenthal equilibrium with strategy $g'(g)$ given by (1) and value functions $V(g)$ and $U(g)$ satisfying equations (2) and (8). ■

# 2 Proof of Proposition 6

The strategy of the proof is identical to that of the proof of Proposition 1. We first derive the bureaucrat value function implied by the strategy

$$g'(g)=\begin{cases}\frac{(g^*)^2}{(1-\delta)g} & \text{if } g\leq g^*/(1-\delta)\\ (1-\delta)g & \text{if } g>g^*/(1-\delta)\end{cases}. \tag{37}$$

We then show that given this value function, the bureaucrat is always better off choosing according to the strategy (37).

## 2.1 The bureaucrat's value function

There are three cases to consider depending on the current level of the public good $g$. This level could be less than $g^*$, between $g^*$ and $g^*/(1-\delta)$, or larger than $g^*/(1-\delta)$. The second case is easiest because thereafter the public good level remains between $g^*$ and $g^*/(1-\delta)$ and investment is always positive. In particular, letting period 0 denote the current period, period 1 the subsequent period, and so on, we have that $g_0 = g$ and that for all periods $t$ beyond 0, $g_t = (g^*)^2 / [(1-\delta)g_{t-1}]$. This implies that $g_1 = (g^*)^2 / [(1-\delta)g_0]$, $g_2 = g_0$, $g_3 = (g^*)^2 / [(1-\delta)g_0]$, $g_4 = g_0$, etc. Thus

$$U(g) = g + \beta \frac{(g^*)^2}{(1-\delta)g} + \beta^2 g + \beta^3 \frac{(g^*)^2}{(1-\delta)g} + \beta^4 g + ... = \frac{1}{1-\beta^2}\left(g + \beta \frac{(g^*)^2}{(1-\delta)g}\right) \tag{38}$$

Next consider the case in which the current level of the public good is larger than $g^*/(1-\delta)$. In this case, the public good must depreciate before investment is undertaken. As in the proof of Proposition 1, define $\tau(g)$ to be the smallest integer such that $(1-\delta)^\tau g$ is less than or equal to $g^*/(1-\delta)$. Intuitively, starting in period 0 with $g$ units, after $\tau(g)$ periods of no investment the public good level will have dropped below $g^*/(1-\delta)$. Investment will therefore commence in period $\tau(g)$ when the initial level of the public good has fallen to $(1-\delta)^{\tau(g)}g$. Since $(1-\delta)^{\tau(g)}g$ must lie in the interval $[g^*, g^*/(1-\delta)]$, from then on investment will be positive. It follows that for periods $t$ up to and including $\tau(g)$, we have $g_t = (1-\delta)^t g$ and for all subsequent periods $t$, we have $g_t = (g^*)^2 / [(1-\delta)g_{t-1}]$. This implies that when $g$ exceeds $g^*/(1-\delta)$, the bureaucrat's value function is

$$U(g) = \sum_{t=0}^{\tau(g)-1} [\beta(1-\delta)]^t g + \frac{\beta^{\tau(g)}}{1-\beta^2}\left(g(1-\delta)^{\tau(g)} + \beta \frac{(g^*)^2}{g(1-\delta)^{\tau(g)+1}}\right). \tag{39}$$

Finally, consider the case in which the current level of the public good is less than $g^*$. In this case, after investment in period 0, the initial level of the public good in period 1 is $(g^*)^2 / [(1-\delta)g]$ which exceeds $g^*/(1-\delta)$. Period 0 will therefore be followed by a spell with no investment. Investment will not recommence until period $\tau((g^*)^2 / [(1-\delta)g]) + 1$ when the level of the public good has fallen below $g^*/(1-\delta)$. From then on investment will be positive. Thus, we have that for $g \in (0, g^*)$

$$U(g) = g + \beta \left[ \begin{array}{c} \sum_{t=0}^{\tau(\frac{(g^*)^2}{(1-\delta)g})-1} [\beta(1-\delta)]^t \frac{(g^*)^2}{(1-\delta)g} \\ + \frac{\beta^{\tau(\frac{(g^*)^2}{(1-\delta)g})}}{1-\beta^2}\left( \frac{(g^*)^2(1-\delta)^{\tau(\frac{(g^*)^2}{(1-\delta)g})-1}}{g} + \beta \frac{g}{(1-\delta)^{\tau(\frac{(g^*)^2}{(1-\delta)g})}} \right) \end{array} \right]. \tag{40}$$

Combining the three cases, we have the following expression for the bureaucrat's value function:

$$U(g)=\begin{cases} g+\beta\left[\begin{array}{c}\sum_{t=0}^{\tau(\frac{(g^*)^2}{(1-\delta)g})-1}[\beta(1-\delta)]^t\,\frac{(g^*)^2}{(1-\delta)g}\\ +\frac{\beta^{\tau(\frac{(g^*)^2}{(1-\delta)g})}}{1-\beta^2}\left(\frac{(g^*)^2(1-\delta)^{\tau(\frac{(g^*)^2}{(1-\delta)g})-1}}{g}+\beta\frac{g}{(1-\delta)^{\tau(\frac{(g^*)^2}{(1-\delta)g})}}\right)\end{array}\right] & \text{if } g\in(0,g^*)\\ \frac{1}{1-\beta^2}\left(g+\beta\frac{(g^*)^2}{(1-\delta)g}\right) \quad \text{if } g\in[g^*,\frac{g^*}{(1-\delta)}] & \\ \sum_{t=0}^{\tau(g)-1}[\beta(1-\delta)]^t g+\frac{\beta^{\tau(g)}}{1-\beta^2}\left(g(1-\delta)^{\tau(g)}+\beta\frac{(g^*)^2}{g(1-\delta)^{\tau(g)+1}}\right) & \text{if } g>\frac{g^*}{(1-\delta)}\end{cases} \tag{41}$$

With this expression in hand, we can now establish the following useful result:

**Lemma 2** *The bureaucrat's value function as defined in (41) is continuous. On the interval* $(0,g^*)$*, depending on the parameters* $\beta$ *and* $\delta$*, the value function is either decreasing or first decreasing and then increasing. On the interval* $[g^*,\infty)$*, the value function is increasing.*

**Proof of Lemma 2** We begin by analyzing the behavior of the value function (41) on the interval $[g^*,\frac{g^*}{(1-\delta)}]$. Note that in the interior of this range, $U(g)$ is differentiable with derivative

$$U'(g)=\frac{1}{1-\beta^2}\left(1-\beta\frac{(g^*)^2}{(1-\delta)g^2}\right)\geq\frac{1}{1-\beta^2}\left(1-\frac{\beta}{(1-\delta)}\right)>0.$$

Thus, the bureaucrat's utility is continuous on the interior of this interval and increasing.

Next consider the behavior of the value function on the interval $(\frac{g^*}{(1-\delta)},\infty)$. Note that we can partition this interval as follows:

$$(\frac{g^*}{(1-\delta)},\infty)=\cup_{t=1}^{\infty}(\frac{g^*}{(1-\delta)^t},\frac{g^*}{(1-\delta)^{t+1}}].$$

Moreover, for any $g\in(\frac{g^*}{(1-\delta)^t},\frac{g^*}{(1-\delta)^{t+1}}]$ it is the case that $\tau(g)=t$. Thus, on the interval $(\frac{g^*}{(1-\delta)^t},\frac{g^*}{(1-\delta)^{t+1}}]$ we have

$$U(g)=\sum_{\tau=0}^{t-1}[\beta(1-\delta)]^\tau g+\frac{\beta^t}{1-\beta^2}\left(g(1-\delta)^t+\beta\frac{(g^*)^2}{g(1-\delta)^{t+1}}\right).$$

It follows that, on the interior of this interval, $U(g)$ is differentiable with derivative

$$\begin{aligned}U'(g) &= \sum_{\tau=0}^{t-1}[\beta(1-\delta)]^\tau+\frac{\beta^t}{1-\beta^2}\left((1-\delta)^t-\beta\frac{(g^*)^2}{g^2(1-\delta)^{t+1}}\right)\\ &\geq \sum_{\tau=0}^{t-1}[\beta(1-\delta)]^\tau+\frac{\beta^t(1-\delta)^t}{1-\beta^2}\left(1-\frac{\beta}{1-\delta}\right)>0,\end{aligned}$$

Thus, the bureaucrat's utility is continuous on the interior of this interval and increasing. Moreover, note that

$$\begin{aligned}
\lim_{g \searrow \frac{g^*}{(1-\delta)}} U(g) &= \frac{g^*}{(1-\delta)} + \frac{\beta}{1-\beta^2}\left(g^* + \frac{\beta}{1-\delta}g^*\right) \\
&= \frac{1}{1-\beta^2}\left(\frac{g^*}{1-\delta} + \beta g^*\right) \\
&= U(\frac{g^*}{(1-\delta)}),
\end{aligned}$$

implying that the value function is continuous at $\frac{g^*}{(1-\delta)}$. Similarly, for any $t \geq 2$

$$\begin{aligned}
\lim_{g \searrow \frac{g^*}{(1-\delta)^t}} U(g) &= \sum_{\tau=0}^{t-1} [\beta(1-\delta)]^\tau \frac{g^*}{(1-\delta)^t} + \frac{\beta^t}{1-\beta^2}\left(g^* + \frac{\beta}{1-\delta}g^*\right) \\
&= \sum_{\tau=0}^{t-2} [\beta(1-\delta)]^\tau \frac{g^*}{(1-\delta)^t} + \beta^{t-1}\frac{g^*}{(1-\delta)} + \frac{\beta^{t-1}}{1-\beta^2}\left(\beta g^* + \frac{\beta^2}{1-\delta}g^*\right) \\
&= U(\frac{g^*}{(1-\delta)^t}).
\end{aligned}$$

Thus, the bureaucrat's value function is continuous at the point $\frac{g^*}{(1-\delta)^t}$. It follows that the bureaucrat's value function is continuous on the interval $(g^*, \infty)$.

Finally, consider the behavior of the value function on the interval $(0, g^*)$. Note first that for any $g \in (0, g^*)$ we can write

$$U(g) = g + \beta U\left(\frac{(g^*)^2}{(1-\delta)g}\right).$$

Since we have already established that $U(\cdot)$ is continuous on the interval $(\frac{g^*}{(1-\delta)}, \infty)$, it follows immediately that the value function is continuous on $(0, g^*)$. Second, note that we can partition the interval $(0, g^*)$ as follows:

$$(0, g^*) = \cup_{t=1}^{\infty}[(1-\delta)^t g^*, (1-\delta)^{t-1}g^*).$$

Moreover, for any $g \in [(1-\delta)^t g^*, (1-\delta)^{t-1}g^*)$ it is the case that $\tau(\frac{(g^*)^2}{(1-\delta)g}) = t$. Thus, on the interval $[(1-\delta)^t g^*, (1-\delta)^{t-1}g^*)$ we have

$$U(g) = g + \beta\left[\sum_{\tau=0}^{t-1}[\beta(1-\delta)]^\tau \frac{(g^*)^2}{(1-\delta)g} + \frac{\beta^t}{1-\beta^2}\left(\frac{(g^*)^2(1-\delta)^{t-1}}{g} + \beta\frac{g}{(1-\delta)^t}\right)\right].$$

It follows that, on the interior of this interval, $U(g)$ is differentiable with derivative

$$U'(g) = 1 + \beta\left[-\sum_{\tau=0}^{t-1}[\beta(1-\delta)]^\tau \frac{(g^*)^2}{(1-\delta)g^2} + \frac{\beta^t}{1-\beta^2}\left(-\frac{(g^*)^2(1-\delta)^{t-1}}{g^2} + \frac{\beta}{(1-\delta)^t}\right)\right]$$

$$= \quad 1+\beta\frac{\beta^t}{1-\beta^2}\left(\frac{\beta}{(1-\delta)^t}\right)-\left(\frac{\beta}{g^2}\right)\left[\sum_{\tau=0}^{t-1}[\beta(1-\delta)]^\tau\frac{(g^*)^2}{1-\delta}+\frac{\beta^t(g^*)^2(1-\delta)^{t-1}}{1-\beta^2}\right].$$

By inspection it is clear that this derivative will be negative for sufficiently small $g$ but otherwise has an ambiguous sign. However, since

$$U''(g)=\left(\frac{\beta}{g^3}\right)\left[\sum_{\tau=0}^{t-1}[\beta(1-\delta)]^\tau\frac{(g^*)^2}{1-\delta}+\frac{\beta^t(g^*)^2(1-\delta)^{t-1}}{1-\beta^2}\right]>0,$$

it is clear that once the bureaucrat's value function starts to increase on the interval $(0,g^*)$ it will continue to do so. Thus, on the interval $(0,g^*)$, depending on the parameters $\beta$ and $\delta$, the value function is either decreasing or first decreasing and then increasing.

It only remains to show that the bureaucrat's value function is continuous at $g^*$. For this observe that on the interval $[(1-\delta)g^*,g^*)$

$$\begin{aligned}\lim_{g\nearrow g^*}U(g) &= g^*+\beta\left[\frac{g^*}{(1-\delta)}+\frac{\beta}{1-\beta^2}\left(g^*+\beta\frac{g^*}{(1-\delta)}\right)\right]\\ &= \frac{1}{1-\beta^2}\left(g^*+\beta\frac{g^*}{(1-\delta)}\right)\\ &= U(g^*).\end{aligned}$$

This completes the proof of Lemma 2. ■

## 2.2 Optimality of the strategy (37)

With the expression for the bureaucrat's value function presented in equation (41) and Lemma 2, it is now possible to show that the bureaucrat always wants to implement the maximum level of investment the citizen will support. We need to show that for any $g$ less than $g^*/(1-\delta)$, $U(g'(g))$ is at least as large as $U(g')$ for any $g'$ in the feasible set $[(1-\delta)g,g'(g)]$. Given that $g'(g)=(g^*)^2/(1-\delta)g$, this feasible set equals $[(1-\delta)g,(g^*)^2/(1-\delta)g]$. By Lemma 2, we know that $U(g')$ is increasing on $[g^*,(g^*)^2/(1-\delta)g]$. We also know that it may be decreasing on at least some part of the interval $[(1-\delta)g,g^*)$ and that, if this is the case, it will be decreasing on the first part of the interval. Thus, it is only necessary to rule out the possibility that $U((1-\delta)g)$ exceeds $U((g^*)^2/[(1-\delta)g])$.

For any $g$ less than $g^*/(1-\delta)$ we have that $(1-\delta)g$ is less than $g^*$. Thus,

$$U((1-\delta)g)=(1-\delta)g+\beta\left[\begin{array}{c}\sum_{t=0}^{\tau(\frac{(g^*)^2}{(1-\delta)^2g})-1}[\beta(1-\delta)]^t\frac{(g^*)^2}{(1-\delta)^2g}+\\ \frac{\beta^{\tau(\frac{(g^*)^2}{(1-\delta)^2g})}}{1-\beta^2}\left(\frac{(g^*)^2(1-\delta)^{\tau(\frac{(g^*)^2}{(1-\delta)^2g})-2}}{g}+\beta\frac{g}{(1-\delta)^{\tau(\frac{(g^*)^2}{(1-\delta)^2g})-1}}\right)\end{array}\right].$$

On the other hand, $(g^*)^2/(1-\delta)g$ is greater than $g^*$ and hence

$$U(\frac{(g^*)^2}{(1-\delta)g}) = \begin{cases} \frac{1}{1-\beta^2}\left(\frac{(g^*)^2}{(1-\delta)g}+\beta g\right) & \text{if } g\in[g^*,\frac{g^*}{(1-\delta)}) \\ \sum_{t=0}^{\tau(\frac{(g^*)^2}{(1-\delta)g})-1}[\beta(1-\delta)]^t\,\frac{(g^*)^2}{(1-\delta)g}+ & \\ \frac{\beta^{\tau(\frac{(g^*)^2}{(1-\delta)g})}}{1-\beta^2}\left(\frac{(g^*)^2}{g}(1-\delta)^{\tau(\frac{(g^*)^2}{(1-\delta)g})-1}+\beta\frac{g}{(1-\delta)^{\tau(\frac{(g^*)^2}{(1-\delta)g})}}\right) & \text{if } g<g^* \end{cases}$$

Suppose first that $g\in[g^*,\frac{g^*}{(1-\delta)})$. We know that

$$\frac{(g^*)^2}{(1-\delta)^2 g}\in(\frac{g^*}{(1-\delta)},\frac{g^*}{(1-\delta)^2}],$$

which implies that $\tau(\frac{(g^*)^2}{(1-\delta)^2 g})=1$. It follows that

$$\begin{aligned} U(\frac{(g^*)^2}{(1-\delta)g})-U((1-\delta)g) &= \frac{1}{1-\beta^2}\left(\frac{(g^*)^2}{(1-\delta)g}+\beta g\right)-(1-\delta)g \\ &\quad -\beta\left[\frac{(g^*)^2}{(1-\delta)^2 g}+\frac{\beta}{1-\beta^2}\left(\frac{(g^*)^2}{(1-\delta)g}+\beta g\right)\right] \\ &= (1-\delta-\beta)\left[\frac{(g^*)^2}{(1-\delta)^2 g}-g\right]\geq 0 \end{aligned}$$

where the latter inequality follows from the assumption that $1-\delta>\beta$ and the fact that $g\leq\frac{g^*}{(1-\delta)}$.

Now consider $g\in[(1-\delta)^t g^*,(1-\delta)^{t-1}g^*)$ for any integer $t\geq 1$. Then we know that

$$\frac{(g^*)^2}{(1-\delta)^2 g}\in(\frac{g^*}{(1-\delta)^{t+1}},\frac{g^*}{(1-\delta)^{t+2}}],$$

which implies that $\tau(\frac{(g^*)^2}{(1-\delta)^2 g})=t+1$. Moreover,

$$\frac{(g^*)^2}{(1-\delta)g}\in(\frac{g^*}{(1-\delta)^t},\frac{g^*}{(1-\delta)^{t+1}}],$$

so that $\tau(\frac{(g^*)^2}{(1-\delta)g})=t$. It follows that

$$\begin{aligned} U(\frac{(g^*)^2}{(1-\delta)g})-U((1-\delta)g) &= \sum_{\tau=0}^{t-1}[\beta(1-\delta)]^\tau\,\frac{(g^*)^2}{(1-\delta)g}+\frac{\beta^t}{1-\beta^2}\left(\frac{(g^*)^2}{g}(1-\delta)^{t-1}+\beta\frac{g}{(1-\delta)^t}\right) \\ &\quad -(1-\delta)g-\beta\left[\begin{array}{c}\sum_{\tau=0}^{t}[\beta(1-\delta)]^\tau\,\frac{(g^*)^2}{(1-\delta)^2 g} \\ +\frac{\beta^{t+1}}{1-\beta^2}\left(\frac{(g^*)^2(1-\delta)^{t-1}}{g}+\beta\frac{g}{(1-\delta)^t}\right)\end{array}\right] \\ &= \beta^t\left(\beta\frac{g}{(1-\delta)^t}\right)+\sum_{\tau=0}^{t}[\beta(1-\delta)]^\tau\,\frac{(g^*)^2}{(1-\delta)g} \end{aligned}$$

$$
\begin{aligned}
& -(1-\delta)g - \beta \sum_{\tau=0}^{t} [\beta(1-\delta)]^{\tau} \frac{(g^*)^2}{(1-\delta)^2 g} \\
= & \sum_{\tau=0}^{t} [\beta(1-\delta)]^{\tau} \frac{(g^*)^2}{(1-\delta)^2 g}(1-\delta-\beta) - \left((1-\delta) - \beta\left(\frac{\beta}{1-\delta}\right)^{t}\right) g \\
\geq & \left[\sum_{\tau=0}^{t} [\beta(1-\delta)]^{\tau} \frac{(1-\delta-\beta)}{(1-\delta)^{t}} - \left((1-\delta)^{t+1} - \beta^{t+1}\right)\right] \frac{g^*}{1-\delta} \\
= & \left[\sum_{\tau=0}^{t} \left(\beta^{\tau}\left(\frac{1}{(1-\delta)}\right)^{t-\tau} - \beta^{\tau}(1-\delta)^{t-\tau}\right)\right] \frac{g^*(1-\delta-\beta)}{1-\delta} \geq 0.
\end{aligned}
$$

Thus, $U((1-\delta)g)$ does not exceed $U((g^*)^2 / [(1-\delta)g])$. ■

## 3 A four period, finite horizon version of the model

Periods are indexed by $t = 1, 2, 3, 4$ and the public good level in period $t$ is denoted $g_t$. The per-period preferences of the bureaucrat and voter are the same as in the paper although we assume throughout that the voter's public good benefits have the quadratic form $B(g_t) = b_0 g_t - b_1 g_t^2$. The depreciation rate of the public good is denoted $\delta$ as before and the players' discount rate is $\beta$. The cost of a unit of the public good is $c$ and the cost of investment in period $t = 1, 2, 3$ is $c\,(g_{t+1} - (1-\delta)g_t)$. No investment occurs in period 4 because investment in period $t$ is not available for consumption until period $t+1$. We assume throughout that $\beta b_0$ exceeds $c$.

To simplify notation, let

$$
\begin{aligned}
\chi_2 &= \frac{\beta b_0 + \beta^2 b_0(1-\delta) + \beta^3 b_0 (1-\delta)^2 - c}{\beta b_1 + \beta^2 b_1 (1-\delta)^2 + \beta^3 b_1 (1-\delta)^4}, \\
\chi_3 &= \frac{\beta b_0 + \beta^2 b_0 (1-\delta) - c}{\beta b_1 + \beta^2 b_1 (1-\delta)^2}, \\
\chi_4 &= \frac{\beta b_0 - c}{\beta b_1},
\end{aligned}
$$

and

$$
\chi^* = \frac{\beta b_0 - c(1-\beta(1-\delta))}{\beta b_1}.
$$

We will make use of the following facts concerning these variables. The proofs are straightforward and thus omitted.

**Fact 1**

$$
\chi_4 < \chi_3 < \chi_2.
$$

**Fact 2**

$$\chi_3(1-\delta) > \chi_4 \Leftrightarrow c\left[\delta + \beta(1-\delta)^2\right] > \delta\beta b_0.$$

**Fact 3**

$$\chi_2(1-\delta) > \chi_3 \Leftrightarrow c\left[\delta + \delta\beta(1-\delta)^2 + \beta^2(1-\delta)^4\right] > \delta\beta b_0\left[1 + \beta(1-\delta)(2-\delta)\right].$$

**Fact 4**

$$\chi_2 < \chi^* \Leftrightarrow \chi_2(1-\delta) > \chi_3.$$

**Fact 5**

$$\chi_3 < \chi^* \Leftrightarrow c\left(1 + \beta(1-\delta)^2\right) > \beta b_0.$$

**Fact 6**

$$\frac{\chi_4}{(1-\delta)} \geq \chi^* \Leftrightarrow \chi_3(1-\delta) < \chi_4.$$

## 3.1 Characterization of equilibrium

We now characterize the sub-game perfect equilibrium of the game defined by the interaction between the voter and the bureaucrat. In standard fashion, we work backwards. As already noted, there is no investment in period 4. Thus, the last period of the interaction is period 3.

### 3.1.1 Period 3

Consider period 3. The current level of the public good is $g_3$. The bureaucrat's choice of investment in period 3 determines the level of public good in period 4. The bureaucrat's problem is:

$$\max_{\{g_4\}}\left\{\begin{array}{c}\beta g_4 \\ s.t.\ \beta\left(b_0 g_4 - b_1 g_4^2\right) - c(g_4 - (1-\delta)g_3) \geq \beta\left[b_0(1-\delta)g_3 - b_1(1-\delta)^2 g_3^2\right]\ \&\ g_4 \geq (1-\delta)g_3\end{array}\right\}.$$

The first constraint ensures that the voter will approve the bureaucrat's proposal and the second rules out disinvestment. Clearly, the bureaucrat will want to make $g_4$ as big as possible. Solving the first constraint and letting $g_4^*(g_3)$ denote the bureaucrat's optimal strategy, this implies

$$g_4^*(g_3) = \left\{\begin{array}{c}\chi_4 - (1-\delta)g_3 \text{ if } \frac{\chi_4}{2(1-\delta)} \geq g_3 \\ (1-\delta)g_3 \text{ if } \frac{\chi_4}{2(1-\delta)} < g_3\end{array}\right., \tag{42}$$

where $\chi_4$ was defined earlier.

### 3.1.2 Period 2

Now consider period 2. The current level of the public good is $g_2$. The bureaucrat's choice of investment in period 2 determines the level of public good in period 3. From our analysis of period 3, the bureaucrat's continuation payoff is

$$U_3(g_3) = \begin{cases} \beta g_3 + \beta^2 \left(\chi_4 - (1-\delta)g_3\right) \text{ if } \frac{\chi_4}{2(1-\delta)} \geq g_3 \\ \beta g_3 + \beta^2(1-\delta)g_3 \text{ if } \frac{\chi_4}{2(1-\delta)} < g_3 \end{cases}.$$

Note that this is increasing in $g_3$, so the bureaucrat will want to choose as large an investment as consistent with voter approval. The voter's continuation payoff if he approves $g_3$ is

$$-c(g_3 - (1-\delta)g_2) + \beta\left[b_0 g_3 - b_1 g_3^2 - c(g_4^*(g_3) - (1-\delta)g_3)\right] + \beta^2\left[b_0 g_4^*(g_3) - b_1 g_4^*(g_3)^2\right].$$

Given what happens in period 3, this equals

$$-c(g_3 - (1-\delta)g_2) + \beta\left[b_0 g_3 - b_1 g_3^2\right] + \beta^2\left[b_0(1-\delta)g_3 - b_1(1-\delta)^2 g_3^2\right].$$

The bureaucrat's problem in period 2 can then be written as

$$\max_{\{g_3\}} \left\{ \begin{array}{c} U_3(g_3) \\ s.t.\ -c(g_3 - (1-\delta)g_2) + \beta\left[b_0 g_3 - b_1 g_3^2\right] + \beta^2\left[b_0(1-\delta)g_3 - b_1(1-\delta)^2 g_3^2\right] \\ \geq \beta\left[b_0(1-\delta)g_2 - b_1(1-\delta)^2 g_2^2\right] + \beta^2\left[b_0(1-\delta)^2 g_2 - b_1(1-\delta)^4 g_2^2\right] \\ \&\ g_3 \geq (1-\delta)g_2 \end{array} \right\}.$$

Solving this problem and denoting the solution by $g_3^*(g_2)$, we get

$$g_3^*(g_2) = \begin{cases} \chi_3 - (1-\delta)g_2 \quad \text{if } \frac{\chi_3}{2(1-\delta)} \geq g_2 \\ (1-\delta)g_2 \quad \text{if } \frac{\chi_3}{2(1-\delta)} < g_2 \end{cases}, \tag{43}$$

where $\chi_3$ was defined earlier.

### 3.1.3 Period 1

Now consider period 1. The current level of the public good is $g_1$. The bureaucrat's choice of investment in period 1 determines the level of public good in period 2. We first understand the bureaucrat's continuation payoff.

**Bureaucrat's continuation payoff** There are two cases to consider.

**Case 1** $\chi_3(1-\delta) < \chi_4$

If the period 2 public good level $g_2$ is such that

$$g_2 < \frac{\chi_3}{(1-\delta)} - \frac{\chi_4}{2(1-\delta)^2},$$

then from our analysis of periods 2 and 3 we may conclude that there will be investment in period 2 but not period 3. The bureaucrat's period 2 continuation payoff is therefore

$$U_2(g_2) = \beta g_2 + \beta^2 \left(\chi_3 - (1-\delta)g_2\right) + \beta^3(1-\delta)\left(\chi_3 - (1-\delta)g_2\right).$$

If

$$g_2 \in \left(\frac{\chi_3}{(1-\delta)} - \frac{\chi_4}{2(1-\delta)^2}, \frac{\chi_3}{2(1-\delta)}\right),$$

then there will be investment in both periods 2 and 3. The bureaucrat's continuation payoff is

$$U_2(g_2) = \beta g_2 + \beta^2 \left(\chi_3 - (1-\delta)g_2\right) + \beta^3 \left(\chi_4 - (1-\delta)\chi_3 + (1-\delta)^2 g_2\right).$$

If

$$g_2 \in \left(\frac{\chi_3}{2(1-\delta)}, \frac{\chi_4}{2(1-\delta)^2}\right),$$

then there will be investment in period 3 but not in period 2. The bureaucrat's continuation payoff is

$$U_2(g_2) = \beta g_2 + \beta^2(1-\delta)g_2 + \beta^3(\chi_4 - (1-\delta)^2 g_2).$$

Finally, if

$$g_2 > \frac{\chi_4}{2(1-\delta)^2},$$

there will be no investment in period 2 or 3. The bureaucrat's continuation payoff is

$$U_2(g_2) = \beta g_2 + \beta^2(1-\delta)g_2 + \beta^3(1-\delta)^2 g_2.$$

Summarizing, the bureaucrat's continuation payoff in Case 1 is

$$U_2(g_2) = \begin{cases} \beta g_2 + \beta^2 \left(\chi_3 - (1-\delta)g_2\right) + \beta^3(1-\delta)\left(\chi_3 - (1-\delta)g_2\right) & g_2 < \frac{\chi_3}{(1-\delta)} - \frac{\chi_4}{2(1-\delta)^2} \\ \beta g_2 + \beta^2 \left(\chi_3 - (1-\delta)g_2\right) + \beta^3 \left(\chi_4 - (1-\delta)\chi_3 + (1-\delta)^2 g_2\right) & g_2 \in \left(\frac{\chi_3}{(1-\delta)} - \frac{\chi_4}{2(1-\delta)^2}, \frac{\chi_3}{2(1-\delta)}\right) \\ \beta g_2 + \beta^2(1-\delta)g_2 + \beta^3(\chi_4 - (1-\delta)^2 g_2) & g_2 \in \left(\frac{\chi_3}{2(1-\delta)}, \frac{\chi_4}{2(1-\delta)^2}\right) \\ \beta g_2 + \beta^2(1-\delta)g_2 + \beta^3(1-\delta)^2 g_2 & g_2 > \frac{\chi_4}{2(1-\delta)^2} \end{cases}. \tag{44}$$

Note for future reference that the bureaucrat's payoff is increasing in $g_2$ except possibly for $g_2$ values less than $\chi_3/(1-\delta) - \chi_4/2(1-\delta)^2$. In this range,

$$U_2'(g_2) = \beta\left[1 - \beta(1-\delta) - \beta^2(1-\delta)^2\right].$$

This is decreasing (increasing) in $g_2$ as 1 is less than (greater than) $\beta(1-\delta) + \beta^2(1-\delta)^2$.

**Case 2** $\chi_3(1-\delta) > \chi_4$

If

$$g_2 < \frac{\chi_3}{2(1-\delta)},$$

then there is investment in period 2 but not period 3. The bureaucrat's continuation payoff is therefore

$$U_2(g_2) = \beta g_2 + \beta^2\left(\chi_3 - (1-\delta)g_2\right) + \beta^3(1-\delta)\left(\chi_3 - (1-\delta)g_2\right).$$

If

$$g_2 > \frac{\chi_3}{2(1-\delta)},$$

then there is no investment in period 2 or 3. The bureaucrat's continuation payoff is therefore

$$U_2(g_2) = \beta g_2 + \beta^2(1-\delta)g_2 + \beta^3(1-\delta)^2 g_2.$$

Summarizing, the bureaucrat's continuation payoff in Case 2 is

$$U_2(g_2) = \begin{cases} \beta g_2 + \beta^2\left(\chi_3 - (1-\delta)g_2\right) + \beta^3(1-\delta)\left(\chi_3 - (1-\delta)g_2\right) & g_2 < \frac{\chi_3}{2(1-\delta)} \\ \beta g_2 + \beta^2(1-\delta)g_2 + \beta^3(1-\delta)^2 g_2 & g_2 > \frac{\chi_3}{2(1-\delta)} \end{cases}. \tag{45}$$

Note for future reference that the bureaucrat's payoff is increasing in $g_2$ except possibly for $g_2$ values less than $\chi_3/2(1-\delta)$. In this range,

$$U_2'(g_2) = \beta\left[1 - \beta(1-\delta) - \beta^2(1-\delta)^2\right].$$

This is decreasing (increasing) in $g_2$ as 1 is less than (greater than) $\beta(1-\delta) + \beta^2(1-\delta)^2$.

**The bureaucrat's constraints** The bureaucrat faces two constraints in selecting $g_2$: voter approval and irreversibility. For the former, note that the voter's continuation payoff if he approves $g_2$ is

$$\begin{aligned} &-c(g_2 - (1-\delta)g_1) + \beta\left[b_0 g_2 - b_1 g_2^2 - c(g_3^*(g_2) - (1-\delta)g_2)\right] \\ &+\beta^2\left[b_0 g_3^*(g_2) - b_1 g_3^*(g_2)^2 - c(g_4^*(g_3) - (1-\delta)g_3^*(g_2))\right] + \beta^3\left[b_0 g_4^*(g_3) - b_1 g_4^*(g_3)^2\right]. \end{aligned}$$

Given what happens in periods 2 and 3, this equals

$$-c(g_2-(1-\delta)g_1)+\beta\left[b_0 g_2 - b_1 g_2^2\right]+\beta^2\left[b_0(1-\delta)g_2 - b_1(1-\delta)^2 g_2^2\right]+\beta^3\left[b_0(1-\delta)^2 g_2 - b_1(1-\delta)^4 g_2^2\right].$$

The constraint of voter approval can therefore be written as:

$$\begin{aligned} & -c(g_2-(1-\delta)g_1)+\beta\left[b_0 g_2 - b_1 g_2^2\right]+\beta^2\left[b_0(1-\delta)g_2 - b_1(1-\delta)^2 g_2^2\right]+\beta^3\left[b_0(1-\delta)^2 g_2 - b_1(1-\delta)^4 g_2^2\right] \\ \geq\ & \beta\left[b_0(1-\delta)g_1 - b_1(1-\delta)^2 g_1^2\right]+\beta^2\left[b_0(1-\delta)^2 g_1 - b_1(1-\delta)^4 g_1^2\right]+\beta^3\left[b_0(1-\delta)^3 g_1 - b_1(1-\delta)^6 g_1^2\right]. \end{aligned}$$

The irreversibility constraint is simply that $g_2 \geq (1-\delta)g_1$. Combining these two constraints, we can write the bureaucrat's constraints as a single constraint

$$g_2 \in \begin{cases} [(1-\delta)g_1, \chi_2-(1-\delta)g_1] \text{ if } g_1 \leq \frac{\chi_2}{2(1-\delta)} \\ \{(1-\delta)g_1\} \text{ if } g_1 > \frac{\chi_2}{2(1-\delta)} \end{cases}, \tag{46}$$

where $\chi_2$ was defined earlier.

**The bureaucrat's optimal strategy** The bureaucrat's optimal strategy solves the problem:

$$\max_{\{g_2\}}\left\{U_2(g_2)\ s.t.\ (46)\right\}.$$

Let $g_2^*(g_1)$ denote the bureaucrat's optimal strategy. The equilibrium is then described by the functions $g_2^*(g_1)$, $g_3^*(g_2)$ and $g_4^*(g_3)$ (where the functions $g_3^*(g_2)$ and $g_4^*(g_3)$ are defined by (43) and (42)). The equilibrium public good levels in the four periods are given by

$$(g_1, g_2^*(g_1), g_3^*(g_2^*(g_1)), g_4^*(g_3^*(g_2^*(g_1)))).$$

The equilibrium is a Romer-Rosenthal equilibrium if and only if

$$g_2^*(g_1) = \chi_2 - (1-\delta)g_1 \text{ if } g_1 \leq \frac{\chi_2}{2(1-\delta)}. \tag{47}$$

Observe that the equilibrium will necessarily be a Romer-Rosenthal equilibrium if 1 is greater than $\beta(1-\delta)+\beta^2(1-\delta)^2$ since this implies that $U_2(g_2)$ is always increasing.

## 3.2 Two questions

While possible, providing a complete description of the equilibrium is tedious because there are so many different cases to consider. In Case 1, there are eight different sub-cases, which one being relevant depending on the exact relationship between the parameters $\{g_1, \chi_2, \chi_3, \chi_4\}$, and in Case 2 there are four different sub-cases. This makes for twelve different sub-cases all together! Accordingly, we will focus the analysis on answering two basic questions. First, will equilibrium always be a Romer-Rosenthal equilibrium? Second, can equilibrium involve the voter approving investment in a period even when the reversion level is at least as big as the level a benevolent planner would choose in that period?

### 3.3 Example 1

We now present an example which answers our first question in the negative. In this example, the equilibrium is not a Romer-Rosenthal equilibrium. We make the following assumptions on the parameters:

$$(\beta, \delta, c, b_0, b_1, g_1) = (0.95, 0.05, 1, 10, 2, 2.64).$$

Under these parameter choices, we have that:

$$(\chi_2, \chi_3, \chi_4) = (5.0372, 4.8381, 4.4737).$$

Observe that $\chi_3(1-\delta) > \chi_4$, so that we are in Case 1. It follows from (45) that the bureaucrat's period 2 continuation value is

$$U_2(g_2) = \begin{cases} (0.95)g_2 + (0.95)^2\,(4.8381 - (0.95)g_2) + (0.95)^3(0.95)\,(4.8381 - (0.95)g_2) & g_2 \leq 2.5464 \\ (0.95)g_2 + (0.5)^2(0.95)g_2 + (0.95)^3(0.95)^2 g_2 & g_2 > 2.5464 \end{cases}.$$

From (46), the bureaucrat's period 2 constraint is

$$g_2 \in [2.508, 2.5292].$$

By inspection, it is clear that $g_2^*(2.64) = 2.508$. Intuitively, the bureaucrat finds it optimal to hold back investment in period 1 to obtain higher public good levels in periods 3 and 4. It is straightforward to show that investment occurs in period 2 and that

$$(g_3^*(g_2^*(2.64)), g_4^*(g_3^*(g_2^*(2.64)))) = (2.455, 2.3327).$$

By contrast, in the Romer-Rosenthal solution investment occurs in periods 1 and 2 and

$$(g_2^*(2.64), g_3^*(g_2^*(2.64)), g_4^*(g_3^*(g_2^*(2.64)))) = (2.5292, 2.4354, 2.3136).$$

### 3.4 Planner's solution

To answer our second question, we must first characterize the public good levels that would be chosen by a planner concerned with maximizing the voter's utility. It will prove most instructive to solve the planner's problem in the same way we solved for the equilibrium - by working backwards from period 3.

#### 3.4.1 Period 3

In period 3 the current level of the public good is $g_3$, and the planner's problem is to choose $g_4$ to solve the problem

$$\max_{\{g_4\}} \left\{ \begin{array}{c} -c(g_4 - (1-\delta)g_3) + \beta(b_0 g_4 - b_1 g_4^2) \\ s.t.\ g_4 \geq (1-\delta)g_3 \end{array} \right\}.$$

Letting the solution be denoted $g_4^o(g_3)$, we have that

$$g_4^o(g_3) = \left\{ \begin{array}{c} \frac{\chi_4}{2} \text{ if } \frac{\chi_4}{2(1-\delta)} \geq g_3 \\ (1-\delta)g_3 \text{ if } \frac{\chi_4}{2(1-\delta)} < g_3 \end{array} \right. . \tag{48}$$

The period 3 continuation value (net of period 3 public good benefits)

$$W_3(g_3) = \left\{ \begin{array}{c} -c(\frac{\chi_4}{2} - (1-\delta)g_3) + \beta(b_0 \frac{\chi_4}{2} - b_1 \left(\frac{\chi_4}{2}\right)^2) \text{ if } \frac{\chi_4}{2(1-\delta)} \geq g_3 \\ \beta(b_0(1-\delta)g_3 - b_1(1-\delta)^2 g_3^2) \text{ if } \frac{\chi_4}{2(1-\delta)} < g_3 \end{array} \right. . \tag{49}$$

Note that

$$W_3'(g_3) = \left\{ \begin{array}{c} c(1-\delta) \ \text{ if } \frac{\chi_4}{2(1-\delta)} \geq g_3 \\ \beta(b_0(1-\delta) - 2b_1(1-\delta)^2 g_3) \ \text{ if } \frac{\chi_4}{2(1-\delta)} < g_3 \end{array} \right. . \tag{50}$$

#### 3.4.2 Period 2

Now consider period 2. The current level of the public good is $g_2$, and the planner's problem is to choose $g_3$ to solve the problem

$$\max_{\{g_3\}} \left\{ \begin{array}{c} -c(g_3 - (1-\delta)g_2) + \beta(b_0 g_3 - b_1 g_3^2 + W_3(g_3)) \\ s.t.\ \ g_3 \geq (1-\delta)g_2 \end{array} \right\}.$$

Let $g_3^o(g_2)$ denote the planner's optimal period 3 level.

Suppose first that the irreversibility constraint does not bind. Then the optimal level of the public good satisfies the first order condition

$$\beta(b_0 - 2b_1 g_3 + W_3'(g_3)) = c.$$

Using (50), if $\frac{\chi_4}{2(1-\delta)} \geq g_3$, then this condition implies

$$g_3 = \frac{\chi^*}{2},$$

where $\chi^*$ was defined earlier. If $\frac{\chi_4}{2(1-\delta)} < g_3$, then the condition implies

$$g_3 = \frac{\chi_3}{2}.$$

Recall from Fact 6 earlier that

$$\frac{\chi_4}{(1-\delta)} > \chi^* \Leftrightarrow \chi_3(1-\delta) < \chi_4.$$

Thus, if the constraint does not bind, the optimal level is $\chi^*/2$ if $\chi_3(1-\delta) < \chi_4$ and $\chi_3/2$ if $\chi_3(1-\delta) \geq \chi_4$. If $\chi_3(1-\delta) < \chi_4$, the constraint binds if

$$g_2 > \frac{\chi^*}{2(1-\delta)},$$

and if $\chi_3(1-\delta) \geq \chi_4$, the constraint binds if

$$g_2 > \frac{\chi_3}{2(1-\delta)}.$$

To summarize, there are two cases.

**Case 1** $\chi_3(1-\delta) < \chi_4$

In this case,

$$g_3^o(g_2) = \begin{cases} \frac{\chi^*}{2} & \text{if } g_2 \leq \frac{\chi^*}{2(1-\delta)} \\ (1-\delta)g_2 & \text{if } g_2 > \frac{\chi^*}{2(1-\delta)} \end{cases}. \tag{51}$$

Using what we know about period 3, the period 2 continuation value (net of period 2 public good benefits)

$$W_2(g_2) = \begin{cases} -c(\frac{\chi^*}{2} - (1-\delta)g_2) + \beta(b_0\frac{\chi^*}{2} - b_1\left(\frac{\chi^*}{2}\right)^2 + c(\frac{\chi_4}{2} - (1-\delta)\frac{\chi^*}{2})) & \\ \quad +\beta^2(b_0\frac{\chi_4}{2} - b_1\left(\frac{\chi_4}{2}\right)^2) & \text{if } g_2 \leq \frac{\chi^*}{2(1-\delta)} \\ \beta(b_0(1-\delta)g_2 - b_1(1-\delta)^2g_2^2 - c(\frac{\chi_4}{2} - (1-\delta)^2g_2)) & \\ \quad +\beta^2(b_0\frac{\chi_4}{2} - b_1\left(\frac{\chi_4}{2}\right)^2) & \text{if } g_2 \in (\frac{\chi^*}{2(1-\delta)}, \frac{\chi_4}{2(1-\delta)^2}) \\ \beta(b_0(1-\delta)g_2 - b_1(1-\delta)^2g_2^2) + \beta^2(b_0(1-\delta)^2g_2 - b_1(1-\delta)^4g_2^2) & \text{if } g_2 \geq \frac{\chi_4}{2(1-\delta)^2} \end{cases} . \quad (52)$$

In the lowest range of $g_2$, values, investment occurs in both periods 3 and 4. In the intermediate range, investment occurs in period 4 but not period 3. In the highest range, there is no investment in either period. Note that in this case

$$W_2'(g_2) = \begin{cases} c(1-\delta) \text{ if } g_2 < \frac{\chi^*}{2(1-\delta)} \\ \beta(b_0(1-\delta) - 2b_1(1-\delta)^2g_2 + c(1-\delta)^2) \text{ if } g_2 \in (\frac{\chi^*}{2(1-\delta)}, \frac{\chi_4}{2(1-\delta)^2}) \\ \beta(b_0(1-\delta) - 2b_1(1-\delta)^2g_2) + \beta^2(b_0(1-\delta)^2 - 2b_1(1-\delta)^4g_2) \text{ if } g_2 > \frac{\chi_4}{2(1-\delta)^2} \end{cases} . \quad (53)$$

**Case 2** $\chi_3(1-\delta) \geq \chi_4$

In this case,

$$g_3^o(g_2) = \begin{cases} \frac{\chi_3}{2} \quad \text{if } g_2 \leq \frac{\chi_3}{2(1-\delta)} \\ (1-\delta)g_2 \quad \text{if } g_2 > \frac{\chi_3}{2(1-\delta)} \end{cases} . \quad (54)$$

Using what we know about period 3, the period 2 continuation value (net of period 2 public good benefits)

$$W_2(g_2) = \begin{cases} -c(\frac{\chi_3}{2} - (1-\delta)g_2) + \beta(b_0\frac{\chi_3}{2} - b_1\left(\frac{\chi_3}{2}\right)^2) & \\ \quad +\beta^2(b_0(1-\delta)\left(\frac{\chi_3}{2}\right) - b_1(1-\delta)^2\left(\frac{\chi_3}{2}\right)^2) & \text{if } g_2 \leq \frac{\chi_3}{2(1-\delta)} \\ \beta(b_0(1-\delta)g_2 - b_1(1-\delta)^2g_2^2) + \beta^2(b_0(1-\delta)^2g_2 - b_1(1-\delta)^4g_2^2) & \text{if } g_2 > \frac{\chi_3}{2(1-\delta)} \end{cases} . \quad (55)$$

In this case, there is no investment in period 3 whatever happens. Investment occurs in period 2 if $g_2 < \chi_3/2(1-\delta)$. Note that in this case

$$W_2'(g_2) = \begin{cases} c(1-\delta) & \text{if } g_2 \leq \frac{\chi_3}{2(1-\delta)} \\ \beta(b_0(1-\delta) - 2b_1(1-\delta)^2 g_2) + \beta^2(b_0(1-\delta)^2 - 2b_1(1-\delta)^4 g_2) & \text{if } g_2 > \frac{\chi_3}{2(1-\delta)} \end{cases}. \tag{56}$$

### 3.4.3 Period 1

Now consider period 1. The current level of the public good is $g_1$, and the planner's problem is to choose $g_2$ to solve the problem

$$\begin{array}{c} \max_{\{g_2\}} -c(g_2 - (1-\delta)g_1) + \beta(b_0 g_2 - b_1 g_2^2 + W_2(g_2)) \\ s.t. \;\; g_2 \geq (1-\delta)g_1 \end{array}.$$

Let $g_2^o(g_1)$ denote the planner's optimal period 2 level. Assuming that the constraint does not bind, the optimal level satisfies the first order condition

$$\beta(b_0 - 2b_1 g_2 + W_2'(g_2)) = c. \tag{57}$$

The planner's solution is described by the functions $g_2^o(g_1)$, $g_3^o(g_2)$ and $g_4^o(g_3)$ (where the functions $g_3^o(g_2)$ and $g_4^o(g_3)$ are defined by (51) or (54) and (42)). The optimal public good levels in the four periods are given by

$$(g_1, g_2^o(g_1), g_3^o(g_2^o(g_1)), g_4^o(g_3^o(g_2^o(g_1)))).$$

Again, there are many different cases to consider as should be clear from (53) and (56). Thus, we do not provide a complete description of the solution here.

## 3.5 Example 2

We now present an example which answers our second question in the affirmative. In this example, equilibrium involves the voter approving investment in period 1 even when the planner would choose not to invest given the initial stock of the public good. Thus, we have that

$$g_2^*(g_1) > (1-\delta)g_1 = g_2^o(g_1).$$

We make the following assumptions on the parameters:

$$(\beta, \delta, c, b_0, b_1, g_1) = (0.6, 0.15, 1, 10, 2, 2.8).$$

Under these parameter choices, we have that:

$$(\chi_2, \chi_3, \chi_4, \chi^*) = (4.9445, 4.6855, 4.1667, 4.5917).$$

Note also that

$$\beta(1-\delta) + \beta^2(1-\delta)^2 = 0.7701 < 1,$$

so that the equilibrium is necessarily a Romer-Rosenthal equilibrium. Since $g_1 \leq \chi_2/2(1-\delta)$ this implies (see (47)) that

$$g_2^*(g_1) = \chi_2 - (1-\delta)g_1 = 2.5645.$$

Turning to the planner's solution, observe that $\chi_3(1-\delta) < \chi_4$, so that we are in Case 2. It follows from (52) that the planner's period 2 continuation value (net of period 2 public good benefits) is

$$W_2(g_2) = \begin{cases} \begin{array}{c} -(2.2959 - (0.85)g_2) + (0.6)(22.959 - 2\,(2.2959)^2 + 2.0834 - (0.85)\,(2.2959)) \\ +(0.6)^2(0.834 - 2\,(2.0834)^2) \end{array} & \text{if } g_2 \leq 2.701 \\ \begin{array}{c} (0.6)((8.5)g_2 - 2(0.85)^2 g_2^2 - (2.0834 - (0.85)^2 g_2)) \\ +(0.6)^2(20.834 - 2\,(2.0834)^2) \end{array} & \text{if } g_2 \in (2.701, 2.8835) \\ (0.6)((8.5)g_2 - 2(0.85)^2 g_2^2) + (0.6)^2(10(0.85)^2 g_2 - 2(0.85)^4 g_2^2) & \text{if } g_2 \geq 2.8835 \end{cases} . \tag{58}$$

We claim that the irreversibility constraint binds so that $g_2^o(g_1) = (1-\delta)g_1 = 2.38$. Suppose, to the contrary, that the irreversibility constraint does not bind. Then $g_2^o(g_1)$ satisfies (57). Given (58), this means that

$$g_2^o(g_1) = 2.2958.$$

But this contradicts the assumption that the irreversibility constraint does not bind. It follows that

$$g_2^*(g_1) = 2.5645 > 2.38 = (1-\delta)g_1 = g_2^o(g_1).$$

Intuitively, the voter approves investment in period 1 in order to reduce public good levels in periods 3 and 4. It is straightforward to show that

$$(g_3^*(g_2^*(g_1)), g_4^*(g_3^*(g_2^*(g_1))) = (2.5057, 2.1298),$$

and that

$$(g_3^o(g_2^o(g_1)), g_4^o(g_3^o(g_2^o(g_1)))) = (2.2959, 2.0834).$$

In the equilibrium, investment occurs in period 2 but not period 3, while in the planner's solution, investment occurs in both periods 2 and 3.